Random Subspaces of a Tensor Product and the Additivity Problem

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joint work with
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- Quantum channels describe the most general physical transformations a quantum system can undergo.

p-norms and p-Rényi entropies

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- The von Neumann entropy is obtained by taking the limit $p \to 1$

Additivity of minimum p-Rényi entropies

Conjecture (Amosov, Holevo, and Werner '00)

For any channels $\mathcal{N}_1, \mathcal{N}_2$, and any $p \geq 1$,

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- Entropies are additive: $H_p(X_1 \otimes X_2) = H_p(X_1) + H_p(X_2)$.
- Given $\mathcal{N}_1, \mathcal{N}_2$, the $\leq$ direction of the equality is trivial, take $X_{12} = X_1 \otimes X_2$.

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- How to get counter-examples:
  1. Lower bound $H_p^{\min}(\mathcal{N}_{1,2})$;
  2. Upper bound $H_p^{\min}(\mathcal{N}_1 \otimes \mathcal{N}_2)$, eg. by finding a particular input X_{12} with low entropy $\rightsquigarrow$ Motohisa's talk;
  3. Conclude by $H_p^{\min}(\mathcal{N}_1 \otimes \mathcal{N}_2) \leq UB < LB_1 + LB_2 \leq H_p^{\min}(\mathcal{N}_1) + H_p^{\min}(\mathcal{N}_2)$.

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- In this talk, we focus on lower bounding the MOE of a quantum channel.

Channels as subspaces

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- Since $\mathcal{N}(P_x) = \mathrm{Tr}_{\text{env}}(WP_xW^*) = \mathrm{Tr}_{\text{env}} P_{Wx}$, the minimal entropies of the channel $\mathcal{N}$ are determined by the image subspace $V = \mathrm{Im} W$.

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- This is a simple question about subspaces of tensor products (equivalently, about the singular values of matrices inside a given subspace V).

Our approach

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- Example: if $d > (k-1)n$, then $(1, 0, \ldots, 0) \in K_V$ and thus $H_p^{\min}(\mathcal{N}) = 0$ for all p.
- We want: subspaces with large MOE and large dimension.

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- From such a radom isometry W, one can construct random quantum channels $\mathcal{N}(X) = \mathrm{Tr}_{\mathsf{n}}(WXW^*)$.
- There are other measures on the Grassmannian one can consider (see Karol's and Motohisa's talks), the one above being the simplest and the most natural.

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Theorem (Belinschi, Collins, N. '10)

For a sequence of uniformly distributed random subspaces V_n, the set K_{V_n} of singular values of unit vectors from V_n converges (almost surely, in the Hausdorff distance) to a deterministic convex subset $K_{k,t}$ of the probability simplex Δ_k

$$K_{k,t} := \{\lambda \in \Delta_k \mid \forall x \in \Delta_k, \langle \lambda, x \rangle \le \|x\|_{(t)}\}.$$

Corollary: exact limit of the MOE

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- However, the set of possible output states is not explicit, and minimizing entropy functions is difficult (work in progress).

How the proof works

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\max_{x \in V, \|x\|=1} \lambda_1(x) &= \max_{x \in V, \|x\|=1} \lambda_1(\mathrm{Tr}_n P_x) \\
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&= \max_{x \in V, \|x\|=1} \max_{y \in \mathbb{C}^k, \|y\|=1} \mathrm{Tr}\left[(\mathrm{Tr}_n P_x) \cdot P_y\right] \\
&= \max_{x \in V, \|x\|=1} \max_{y \in \mathbb{C}^k, \|y\|=1} \mathrm{Tr}\left[P_x \cdot P_y \otimes \mathrm{I}_n\right] \\
&= \max_{y \in \mathbb{C}^k, \|y\|=1} \max_{x \in V, \|x\|=1} \mathrm{Tr}\left[P_x \cdot P_y \otimes \mathrm{I}_n\right] \\
&= \max_{y \in \mathbb{C}^k, \|y\|=1} \|P_V P_y \otimes \mathrm{I}_n\|_\infty .
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- Limit of $\|P_V P_y \otimes \mathrm{I}_n\|_\infty$ for fixed y and random V ?

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- Define $\varphi(\alpha,\beta) = \alpha + \beta - 2\alpha\beta + 2\sqrt{\alpha\beta(1-\alpha)(1-\beta)}$ if $\alpha + \beta \leq 1$ and put $\varphi(\alpha,\beta) = 1$ otherwise.

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Theorem (Collins '05)

In $\mathbb{C}^n$, choose at random according to the Haar measure two independent subspaces V_n and V_n' of respective dimensions $q_n \sim \alpha n$ and $q_n' \sim \beta n$ where $\alpha, \beta \in (0,1)$. Let P_n (resp. P_n') be the orthogonal projection onto V_n (resp. V_n'). Then,

$$\lim_n \|P_n P_n' P_n\|_\infty = \varphi(\alpha,\beta).$$

t-norms

Definition

For a positive integer k, embed $\mathbb{R}^k$ as a self-adjoint real subalgebra $\mathcal{R}$ of a II_1 factor $\mathcal{A}$ endowed with trace τ, so that $\tau((x_1, \ldots, x_k)) = (x_1 + \cdots + x_k)/k$. Let p_t be a projection of rank $t \in (0, 1)$ in $\mathcal{A}$, free from $\mathcal{R}$. On the real vector space $\mathbb{R}^k$, we introduce the following norm, called the (t)-norm:

$$\|x\|_{(t)} := \|p_t x p_t\|_\infty,$$

where the vector $x \in \mathbb{R}^k$ is identified with its image in $\mathcal{R}$.

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Proposition

The distribution $\mu_{t^{-1}p_t x p_t}$ of the (non-commutative) random variable $t^{-1}p_t x p_t$ in the II_1 factor reduced by the projection p_t is

$$\mu_{t^{-1}p_t x p_t} = \mu_x^{\boxplus 1/t}, \quad t \in (0, 1],$$

where $\boxplus$ denotes the free additive convolution of Voiculescu.

The set $K_{k,t}$ and t-norms

- $K_{k,t} := \{\lambda \in \Delta_k \mid \forall x \in \Delta_k, \langle \lambda, x \rangle \leq \|x\|_{(t)}\}$.

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- To get the full result, use $\langle \lambda, x \rangle$ instead of λ_1.
- Unfortunately, it is difficult to compute (t)-norms...

Thank you !

Collins, N. - *Random quantum channels II: Entanglement of random subspaces, Rényi entropy estimates and additivity problems* - Advances in Mathematics 226 (2011), 1181-1201.

Belinschi, Collins, N. - *Laws of large numbers for eigenvectors and eigenvalues associated to random subspaces in a tensor product* - to appear in Inventiones Mathematicae.