Random Subspaces of a Tensor Product and the Additivity Problem

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joint work with
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Eigen- and singular values

- Singular value decomposition of a matrix $X \in \mathcal{M}_{k\times n}(\mathbb{C})$

$$X = \sum \sqrt{\lambda_i(X)} e_i \otimes f_i^*,$$

where e_i, f_i are orthonormal families in $\mathbb{C}^k$, $\mathbb{C}^n$, and $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq 0$ are the singular values of X.
- The eigenvalues of the matrix XX^* are $\lambda_i(x)$.
- Question: What are the singular values of a random matrix ?

Singular values of random matrices, k fixed, $n \to \infty$ regime

- Let X be a $k \times n$ Ginibre random matrix, i.e. $\{X_{ij}\}$ are i.i.d. complex Gaussian random variables.
- We are interested in long matrices: k fixed, $n \to \infty$.
- We normalize our matrices, by taking them on the unit Euclidean sphere $\mathrm{Tr} XX^* = 1$.
- Thus, the singular values vector $\lambda(X)$ is a probability vector

$$\lambda(X) \in \Delta_k^{\downarrow} = \{y \in \mathbb{R}^k \,:\, y_i \geq 0, \sum_i y_i = 1, \, y_1 \geq \cdots \geq y_k\}.$$

- It is an easy exercise to show that, almost surely,

$$\forall i, \qquad \lambda_i(X) \to 1/k.$$

Vector formulation

- Recall: SVD of $X \in \mathcal{M}_{k\times n}(\mathbb{C})$

$$X = \sum \sqrt{\lambda_i(X)} e_i \otimes f_i^*.$$

- Using the isomorphism $\mathbb{C}^k \otimes C^n \simeq \mathcal{M}_{k\times n}(\mathbb{C})$, X can be seen as a vector in a tensor product $x \in \mathbb{C}^k \otimes \mathbb{C}^n$.
- The vector x admits a Schmidt decomposition $x = \sum_i \sqrt{\lambda_i(x)} e_i \otimes f_i$.
- The eigenvalues of the matrix $XX^* = [\mathrm{id}_k \otimes \mathrm{Tr}_n] P_x$ are $\lambda_i(x)$.
- Problem: What are the singular values of **ALL** vectors [matrices] inside a (random) subspace V of a tensor product [matrix space] ?
- This is a simple question about subspaces of tensor products (equivalently, about the singular values of matrices inside a given subspace V).

Singular values of vectors from a subspace

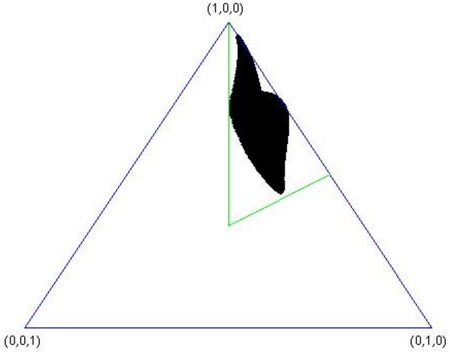
- For a subspace $V \subset \mathbb{C}^k \otimes \mathbb{C}^n$ of dimension d, define the set eigen-/singular values or Schmidt coefficients

$$K_V = \{\lambda(x) : x \in V, \|x\| = 1\}.$$

- Our goal is to understand K_V.
- The set K_V is a compact subset of the ordered probability simplex $\Delta_k^{\downarrow}$.
- Local invariance: $K_{(U_1 \otimes U_2)V} = K_V$, for unitary matrices $U_1 \in \mathcal{U}(k)$ and $U_2 \in \mathcal{U}(n)$.
- Monotonicity: if $V_1 \subset V_2$, then $K_{V_1} \subset K_{V_2}$.
- Example: $d = 1$, $V = \mathbb{C}x$. We have $K_V = \{\lambda(x)\}$.
- Example: if $d > (k-1)(n-1)$, then $(1, 0, \ldots, 0) \in K_V$.

Examples

- : $V = \mathrm{span}\{G_1, G_2\}$, where $G_{1,2}$ are 3×3 independent Ginibre random matrices.



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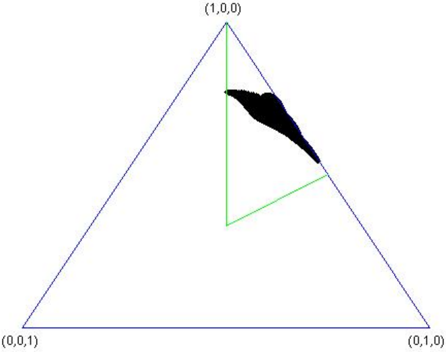

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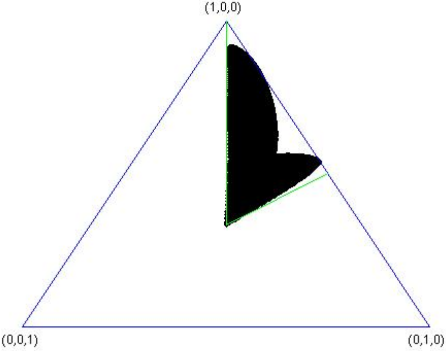
- : $V = \mathrm{span}\{\mathrm{I}_3, G\}$, where G is a 3×3 Ginibre random matrix.



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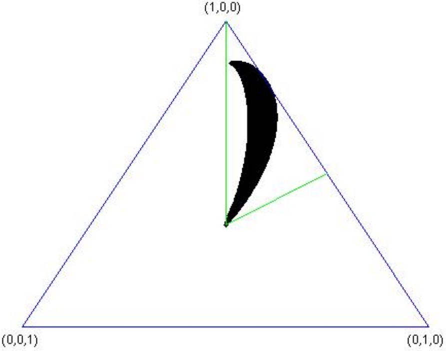

Why do we care ? Quantum channels !

- Quantum states with d degrees of freedom are described by density matrices

$$X \in \mathcal{M}^{\mathsf{sa}}(\mathbb{C}^d); \qquad \mathrm{Tr} X = 1 \text{ and } X \geq 0.$$

- Quantum channels $\mathcal{N} : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^k)$ are completely positive, trace-preserving maps. In particular, they send quantum states to quantum states.
  - Complete positivity CP: $\mathcal{N} \otimes \mathrm{id}_n$ preserves positivity.
  - Trace preservation TP: $\mathrm{Tr}[\mathcal{N}(X)] = \mathrm{Tr}(X)$ for all X.
- Quantum channels describe the most general physical transformations a quantum system can undergo.

The additivity problem

- The von Neumann entropy of $X \in \mathcal{M}^{1,+}(\mathbb{C}^d)$

$$H(X) = -\mathrm{Tr}(X \log X).$$

- The entropy is additive: $H(X_1 \otimes X_2) = H(X_1) + H(X_2)$.
- The minimum output entropy of a quantum channel is

$$H^{\min}(\mathcal{N}) = \min_{X \in \mathcal{M}^{1,+}(\mathbb{C}^d)} H(\mathcal{N}(X)).$$

Conjecture (Amosov, Holevo, and Werner '00)

For any channels $\mathcal{N}_1, \mathcal{N}_2$

$$H^{min}(\mathcal{N}_1 \otimes \mathcal{N}_2) = H^{min}(\mathcal{N}_1) + H^{min}(\mathcal{N}_2).$$

- Given $\mathcal{N}_1, \mathcal{N}_2$, the $\leq$ direction of the equality is trivial, take $X_{12} = X_1 \otimes X_2$.

An important problem

- Additivity of MOE is equivalent to other additivity conjectures in quantum information theory [Shor '03], such as the additivity of Holevo capacity for quantum channels or the additivity of entanglement of formation for quantum bipartite states.
- Additivity has been shown to hold for a large class of channels: unitary, unital qubit, depolarizing, dephasing, entanglement breaking, ...
- But ... the Additivity Conjecture is false!
- How to get counter-examples:
  1. Lower bound $H^{\min}(\mathcal{N}_{1,2})$;
  2. Upper bound $H^{\min}(\mathcal{N}_1 \otimes \mathcal{N}_2)$, eg. by finding a particular input X_{12} with low entropy;
  3. Conclude by $H^{\min}(\mathcal{N}_1 \otimes \mathcal{N}_2) \leq UB < LB_1 + LB_2 \leq H^{\min}(\mathcal{N}_1) + H^{\min}(\mathcal{N}_2)$.
- In this talk, we focus on lower bounding the MOE of a quantum channel.

Channels as subspaces

- Let $V \subset \mathbb{C}^k \otimes \mathbb{C}^n$ be a subspace of dimension d and consider an isometry $W : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n$ with $\mathrm{Im}\, W = V$.
- One can define a channel $\mathcal{N} : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^k)$ by

$$\mathcal{N}(X) = [\mathrm{id}_\mathsf{k} \otimes \mathrm{Tr}_\mathsf{n}](WXW^*).$$

- Every channel can be defined in this way (by choosing n large enough).
- By convexity properties, the MOE is attained on pure states i.e. rank one projectors.
- Since $\mathcal{N}(P_x) = [\mathrm{id}_\mathsf{k} \otimes \mathrm{Tr}_\mathsf{n}](WP_xW^*) = [\mathrm{id}_k \otimes \mathrm{Tr}_n]P_{Wx}$, the minimal entropies of the channel $\mathcal{N}$ are determined by the image subspace $V = \mathrm{Im}\, W$.

Random subspaces

- Idea: when you do not know how to find a subspace having some nice properties, pick one at random!
- There is an uniform (or Haar) measure on the set of d-dimensional subspaces of $\mathbb{C}^{kn}$.
- Take a $kn \times kn$ Haar distributed random unitary matrix $U \in \mathcal{U}(kn)$ and take V to be the span of its first d columns.
- Alternatively, if W is a $kn \times d$ truncation of U, then $V = \mathrm{Im} W$ is uniform.
- From such a radom isometry W, one can construct random quantum channels $\mathcal{N}(X) = [\mathrm{id}_k \otimes \mathrm{Tr}_n](WXW^*)$.
- There are other measures on the Grassmannian one can consider, the one above being the simplest and the most natural.

Main result

- For the rest of the talk, we consider the following asymptotic regime: k fixed, $n \to \infty$, and $d \sim tkn$, for a fixed parameter $t \in (0, 1)$.

Theorem (Belinschi, Collins, N. '10)

For a sequence of uniformly distributed random subspaces V_n, the set K_{V_n} of singular values of unit vectors from V_n converges (almost surely, in the Hausdorff distance) to a deterministic convex subset $K_{k,t}$ of the probability simplex Δ_k

$$K_{k,t} := \{\lambda \in \Delta_k \mid \forall x \in \Delta_k, \langle \lambda, x \rangle \leq \|x\|_{(t)}\}.$$

Corollary: exact limit of the MOE

- By the previous theorem, in the specific asymptotic regime t, k fixed, $n \to \infty$, $d \sim tkn$, we have the following a.s. convergence result for random quantum channels:

$$\lim_{n\to\infty} H^{\min}(\mathcal{N}) = \min_{\lambda \in K_{k,t}} H(\lambda).$$

- It is not just a bound, the exact limit value is obtained ☺
- However, the set $K_{k,t}$ is not explicit, and minimizing entropy functions is difficult ☹

Idea of the proof

- Question: what is the maximum singular value $\max_{x \in V, \|x\|=1} \lambda_1(x)$ of a unit vector from V ?
- Compute

$$\begin{aligned}
\max_{x \in V, \|x\|=1} \lambda_1(x) &= \max_{x \in V, \|x\|=1} \lambda_1([\mathrm{id}_k \otimes \mathrm{Tr}_n] P_x) \\
&= \max_{x \in V, \|x\|=1} \|[\mathrm{id}_k \otimes \mathrm{Tr}_n] P_x\| \\
&= \max_{x \in V, \|x\|=1} \max_{y \in \mathbb{C}^k, \|y\|=1} \mathrm{Tr}\left[([\mathrm{id}_k \otimes \mathrm{Tr}_n] P_x) \cdot P_y\right] \\
&= \max_{x \in V, \|x\|=1} \max_{y \in \mathbb{C}^k, \|y\|=1} \mathrm{Tr}\left[P_x \cdot P_y \otimes \mathrm{I}_n\right] \\
&= \max_{y \in \mathbb{C}^k, \|y\|=1} \max_{x \in V, \|x\|=1} \mathrm{Tr}\left[P_x \cdot P_y \otimes \mathrm{I}_n\right] \\
&= \max_{y \in \mathbb{C}^k, \|y\|=1} \|P_V \cdot P_y \otimes \mathrm{I}_n\|_\infty .
\end{aligned}$$

- Limit of $\|P_V \cdot P_y \otimes \mathrm{I}_n\|_\infty$ for fixed y and random V ?

Theorem (Collins '05)

In $\mathbb{C}^n$, choose at random according to the Haar measure two independent subspaces V_n and V'_n of respective dimensions $q_n \sim \alpha n$ and $q'_n \sim \beta n$ where $\alpha, \beta \in (0,1)$. Let P_n (resp. P'_n) be the orthogonal projection onto V_n (resp. V'_n). Then,

$$\lim_n \|P_n P'_n P_n\|_\infty = \varphi(\alpha, \beta).$$

- One can compute $\varphi(\alpha, \beta) = \alpha + \beta - 2\alpha\beta + 2\sqrt{\alpha\beta(1-\alpha)(1-\beta)}$ if $\alpha + \beta \leq 1$ and $\varphi(\alpha, \beta) = 1$ if $\alpha + \beta > 1$ (subspaces V_n and V'_n have non-trivial intersection).

t-norms

Definition

For a positive integer k, embed $\mathbb{R}^k$ as a self-adjoint real subalgebra $\mathcal{R}$ of a II_1 factor $(\mathcal{A}, \tau)$, so that $\tau(x) = (x_1 + \cdots + x_k)/k$. Let p_t be a projection of rank $t \in (0,1)$ in $\mathcal{A}$, free from $\mathcal{R}$. On the real vector space $\mathbb{R}^k$, we introduce the following norm, called the (t)-norm:

$$\|x\|_{(t)} := \|p_t x p_t\|_\infty,$$

where the vector $x \in \mathbb{R}^k$ is identified with its image in $\mathcal{R}$.

Proposition

The distribution $\mu_{t^{-1}p_t x p_t}$ of the (non-commutative) random variable $t^{-1}p_t x p_t$ in the II_1 factor reduced by the projection p_t is $\mu_{t^{-1}p_t x p_t} = \mu_x^{\boxplus 1/t}, \quad t \in (0,1]$, where $\boxplus$ denotes the free additive convolution of Voiculescu.

The set $K_{k,t}$ and t-norms

- $K_{k,t} := \{\lambda \in \Delta_k \mid \forall x \in \Delta_k, \langle \lambda, x\rangle \le \|x\|_{(t)}\}$.
- Recall that

$$\max_{x\in V, \|x\|=1} \lambda_1(x) = \max_{y\in\mathbb{C}^k, \|y\|=1} \|P_V P_y \otimes \mathrm{I}_n\|_\infty.$$

- For fixed y, P_V and $P_y \otimes \mathrm{I}_n$ are independent projectors of relative ranks t and $1/k$ respectively.
- Thus, $\|P_V \cdot P_y \otimes \mathrm{I}_n\|_\infty \to \varphi(t, 1/k) = \|(1, 0, \ldots, 0)\|_{(t)}$.
- We can take the max over y at no cost, by considering a finite net of y's, since k is fixed.
- To get the full result, use $\langle \lambda, x\rangle$ (for all directions x) instead of λ_1.
- Unfortunately, it is difficult to compute (t)-norms, so we do not have an explicit formula for $K_{k,t}$.

Thank you !

Collins, N. - *Random quantum channels II: Entanglement of random subspaces, Rényi entropy estimates and additivity problems*.

Belinschi, Collins, N. - *Laws of large numbers for eigenvectors and eigenvalues associated to random subspaces in a tensor product*.