# Positive and completely positive maps via free additive powers of probability measures

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*joint work with Benoit Collins (uOttawa) and Patrick Hayden (McGill)*

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# Entanglement in Quantum Information Theory

- Quantum states with $n$ degrees of freedom are described by density matrices

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- For rank one quantum states, entanglement can be detected and quantified by the entropy of entanglement

$$E_{\mathrm{ent}}(P_x) = H(s(x)) = -\sum_{i=1}^{\min(m,n)} s_i(x) \log s_i(x),$$

where $x \in \mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{M}_{m \times n}(\mathbb{C})$ is seen as a $m \times n$ matrix and $s_i(x)$ are its singular values.

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- A pure state $x \in \mathbb{C}^m \otimes \mathbb{C}^n$ is separable $\iff$ $E_{\mathrm{ent}}(P_x) = 0$.

# An image of entanglement

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## Shor's quantum factoring algorithm

- Runs on a quantum computer with polynomial time $O(\log^3 N)$.
- Classical sieve algorithms run in sub-exponential time $O(\exp(\log^{1/3} N))$.
- Entanglement is necessary for the exponential speed-up.
- State of the art factorization in labs: $21 = 3 \times 7$ [2011], 143 (?) [2012].

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- Detecting entanglement for general states in low dimension $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$ is possible via the PPT criterion [Horodecki].
- In general, there exists a countable hierarchy of conditions characterizing separability [Doherty et al] that can be checked by semidefinite programming.

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- Let $\mathcal{N} : \mathbb{M}_n \to \mathcal{A}$ be a positive map. Then, for every separable state $\rho_{12} \in \mathbb{M}_{mn}^{1,+}$, one has $[\mathrm{id}_m \otimes \mathcal{N}](\rho_{12}) \geqslant 0$.
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- Hence, positive, but not CP maps $\mathcal{N}$ provide sufficient entanglement criteria: if $[\mathrm{id}_m \otimes \mathcal{N}](\rho_{12}) \ngeqslant 0$, then $\rho_{12}$ is entangled.

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- Hence, positive, but not CP maps $\mathcal{N}$ provide sufficient entanglement criteria: if $[\mathrm{id}_m \otimes \mathcal{N}](\rho_{12}) \ngeqslant 0$, then $\rho_{12}$ is entangled.
- Moreover, if $[\mathrm{id}_m \otimes \mathcal{N}](\rho_{12}) \geqslant 0$ for all positive, but not CP maps $\mathcal{N}$, then $\rho_{12}$ is separable.

# Positive Partial Transpose matrices

- The transposition map $\mathrm{t}$ is positive, but not CP. Define the convex set

$$\mathcal{PPT} = \{\rho_{12} \in \mathbb{M}_{mn}^{1,+} \mid [\mathrm{id}_m \otimes \mathrm{t}_n](\rho_{12}) \geqslant 0\}.$$

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- For $(m, n) \in \{(2,2),(2,3)\}$ we have $\mathcal{SEP} = \mathcal{PPT}$. In other dimensions, the inclusion $\mathcal{SEP} \subset \mathcal{PPT}$ is strict.

# The PPT criterion at work

- Recall the Bell state $\rho_{12} = P_{\mathrm{Bell}}$, where

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$$\rho_{12} = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) = \frac{1}{2}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

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- Partial transposition: transpose each block $B_{ij}$:

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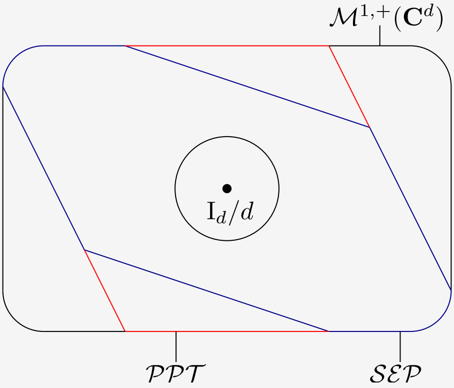
- This matrix is no longer positive $\implies$ the state is entangled.

# Three convex sets



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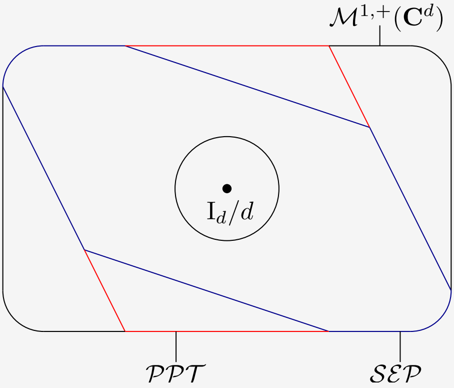


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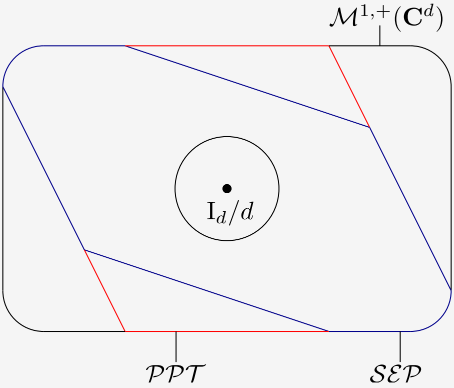


- States in $\mathcal{PPT} \setminus \mathcal{SEP}$ are called bound entangled: no "maximal" entangled can be distilled from them.
- All these sets contain an open ball around the identity.

# The Choi matrix of a map

- For any $n$, recall that the maximally entangled state is the orthogonal projection onto

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- To any map $\mathcal{N} : \mathbb{M}_n \to \mathcal{A}$, associate its Choi matrix

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**Theorem (Choi '72)**

*A map $\mathcal{N} : \mathbb{M}_n \to \mathcal{A}$ is CP iff its Choi matrix $C_{\mathcal{N}}$ is positive.*

# The Choi-Jamiolkowski isomorphism

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- The map $\mathcal{N} \mapsto C_{\mathcal{N}}$ is called the Choi-Jamiolkowski isomorphism.
- It sends:
  1. All linear maps to all operators;
  2. Hermicity preserving maps to hermitian operators;
  3. Entanglement breaking maps to separable quantum states;
  4. Unital maps to operators with unit left partial trace ($[\mathrm{Tr} \otimes \mathrm{id}]C_{\mathcal{N}} = \mathrm{I}_d$);
  5. Trace preserving maps to operators with unit left partial trace ($[\mathrm{id} \otimes \mathrm{Tr}]C_{\mathcal{N}} = \mathrm{I}_n$).

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**Theorem**

*A map $\mathcal{N} : \mathbb{M}_n \to \mathcal{A}$ is $k$-positive iff its Choi matrix $C_{\mathcal{N}}$ is $k$-positive. In particular, $\mathcal{N}$ is positive iff $C_{\mathcal{N}}$ is block-positive.*

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- When $d \to \infty$, the spectrum of $A_d + B_d$ converges to $a \boxplus b$.

# Proof ingredients

Let $\mathcal{N}_\mu^{(d)} : \mathbb{M}_n \to \mathbb{M}_d$ be the map whose Choi matrix is $U_d X_d U_d^*$.

## Theorem

*If $\operatorname{supp}(\mu^{\boxplus n/k}) \subset (0, \infty)$ then, almost surely as $d \to \infty$, the map $\mathcal{N}_\mu^{(d)}$ is $k$-positive. If $\operatorname{supp}(\mu^{\boxplus n/k}) \cap (-\infty, 0) \neq \emptyset$ then, almost surely as $d \to \infty$, $\mathcal{N}_\mu^{(d)}$ is not $k$-positive.*

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## Proposition

*A map $\mathcal{N}$ is $k$-positive iff for any self-adjoint projection $P \in \mathbb{M}_n$ of rank $k$, the operator $P \otimes \mathrm{I}_\mathcal{A} \cdot C_\mathcal{N} \cdot P \otimes 1_\mathcal{A}$ is positive.*

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## Proposition (Nica and Speicher)

*Let* $x, p$ *be free elements in a ncps* $(M, \tau)$ *and assume that* $p$ *is a selfadjoint projection of rank* $\tau(p) = 1/t$ $(t \geqslant 1)$ *and that* $x$ *has distribution* $\mu$*. Then, the distribution of* $t^{-1}pxp$ *inside the contracted ncps* $(pMp, \tau(p \cdot p))$ *is* $\mu^{\boxplus t}$

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**Theorem**

*The map $\mathcal{N}_\mu$ is k-positive iff* $\operatorname{supp}(\mu^{\boxplus n/k}) \subseteq \mathbb{R}_+$.

# Example: semicircular measures

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- In free probability theory, $s_{0,1}$ plays the role of the standard Gaussian in classical probability, cf Free Central Limit Theorem.
- We have $s_{a,\sigma}^{\boxplus n/k} = s_{an/k,\sigma\sqrt{n/k}}$, with support $\mathrm{supp}(s_{a,\sigma}^{\boxplus n/k}) = [an/k - 2\sigma\sqrt{n/k}, an/k + 2\sigma\sqrt{n/k}]$.

## Theorem

*Let $n$ be an integer and $a, \sigma$ some positive parameters. The map $\mathcal{N}_{a,\sigma} : \mathbb{M}_n \to M$ associated to a semi-circular distribution $s_{a,\sigma}$ is $k$-positive iff $k \leqslant 4n\sigma/a^2$. In particular, for any $n$ and any $k < n$, there exist parameters $a, \sigma > 0$ such that the above map is $k$-positive but not $k+1$-positive.*

Merci !