Random Subspaces of a Tensor Product and the Additivity Problem

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joint work with
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Outline of the talk

1. The set of singular values of a vector subspace
   - Singular values of random matrices
   - Tensor formulation
   - The set K_V
2. Quantum channels and additivity problems
   - States and channels
   - The additivity problem
   - Random quantum channels
3. Almost sure limit for K_V and free probability
   - Statement of the main result and applications
   - Free probability - a review
   - Sketch of the proof

Eigen- and singular values

Singular value decomposition of a matrix $X \in \mathbb{M}_{k \times n}(\mathbb{C})$ ($k \leq n$)

$$X = U\Sigma V^* = \sum_{i=1}^{k} \sqrt{\lambda_i(XX^*)} e_i f_i^*,$$

where e_i, f_i are orthonormal families in $\mathbb{C}^k$, $\mathbb{C}^n$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ are the (squares of the) singular values of X, or the eigenvalues of XX^*.

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Question

What are the singular values of a generic/random matrix ?

Singular values of random matrices, M-P regime

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Theorem (Marchenko, Pastur '67)

Almost surely, the empirical singular value distribution

$$\mu_X = \frac{1}{k}\sum_{i=1}^{k} \delta_{\lambda_i(n^{-1}XX^*)}$$

converges weakly to the Marchenko-Pastur distribution of parameter c:

$$a.s. \quad \mu_X \to \pi_c.$$

The Marchenko-Pastur distribution

The Marchenko-Pastur distribution of parameter $c \in (0, \infty)$ is

$$\pi_c = \max(1 - c, 0)\delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x}\mathbf{1}_{[a,b]}(x)dx,$$

where $a = (\sqrt{c} - 1)^2$ and $b = (\sqrt{c} + 1)^2$.

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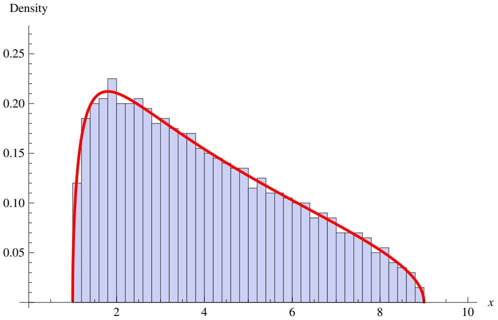
where $a = (\sqrt{c} - 1)^2$ and $b = (\sqrt{c} + 1)^2$. In the figure, the density for $c = 4$ (red) is plotted, along with the singular values of a 1000×4000 random matrix.



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Thus, the singular values vector $\lambda(XX^*)$ is a probability vector

$$\lambda(XX^*) \in \Delta_k^{\downarrow} = \{y \in \mathbb{R}^k \,:\, y_i \geq 0, \sum_i y_i = 1, \, y_1 \geq \cdots \geq y_k\}.$$

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It is an easy exercise to show that, almost surely,

$$\forall i = 1, 2, \ldots, k \,, \qquad \lambda_i(XX^*) \to 1/k.$$

From matrices to 2-tensors

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Question

What are the singular values of ALL vectors [matrices] inside a (random) subspace V of a tensor product [matrix space] ?

Singular values of vectors from a subspace

For a subspace $V \subset \mathbb{C}^k \otimes \mathbb{C}^n$ of dimension $\dim V = d$, define the set eigen-/singular values or Schmidt coefficients

$$K_V = \{\lambda(x) : x \in V, \|x\| = 1\}.$$

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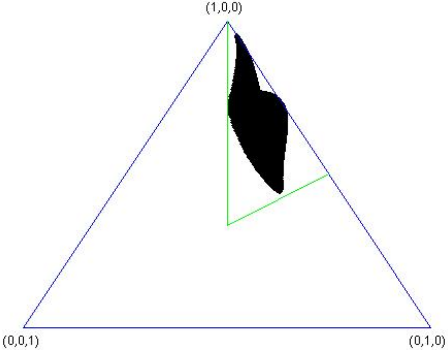
- Monotonicity: if $V_1 \subset V_2$, then $K_{V_1} \subset K_{V_2}$.

Examples

$V = \mathrm{span}\{G_1, G_2\}$, where $G_{1,2}$ are 3×3 independent Ginibre random matrices.



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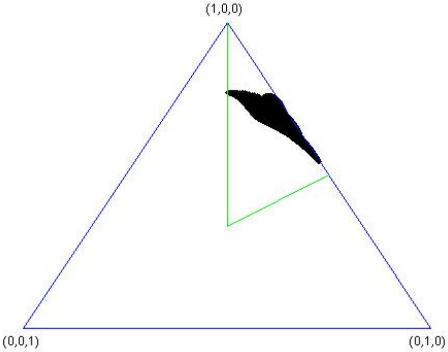

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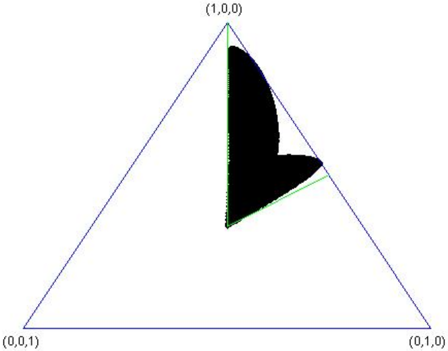

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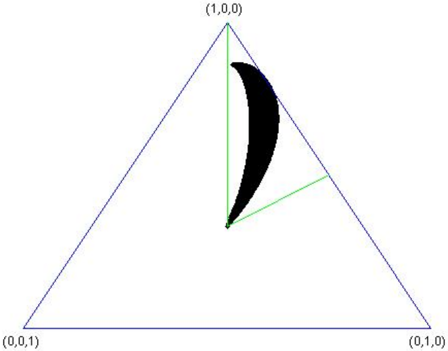

States and channels in quantum information theory

Quantum states with n degrees of freedom are described by density matrices

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Quantum channels describe the most general physical transformations a quantum system can undergo.

The (minimum output) von Neumann entropy

- The Shannon entropy of a probability vector $\lambda \in \Delta_k$

$$H(\lambda) = -\sum_{i=1}^{k} \lambda_i \log \lambda_i \in [0, \log k].$$

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- The minimum output entropy of a quantum channel F is

$$\boxed{H^{\min}(F) = \min_{X \in \mathbb{M}_d^{1,+}} H(F(X)).}$$

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Conjecture (Amosov, Holevo and Werner '00)

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- But... the Additivity Conjecture is false ! [Hayden, Winter '08, Hastings '09]
- Counterexamples: random channels.

Random channels via random isometries

- Consider an isometry $W : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n$ and let V be its image $\mathrm{Im} W = V$ so that $V \subset \mathbb{C}^k \otimes \mathbb{C}^n$ is a subspace of dimension d.

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- There is an uniform (or Haar) measure on the set of isometries $\{W : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n \ : \ WW^* = \mathrm{I}_d\}$: take a $kn \times kn$ Haar distributed random unitary matrix $U \in \mathcal{U}(kn)$ and take W to be the restriction of U to the first d coordinates.

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- We call random quantum channels the probability distribution obtained as the push-forward of this measure through the Stinespring dilation.

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- Since $F(P_x) = [\mathrm{id_k} \otimes \mathrm{Tr_n}](WP_xW^*) = [\mathrm{id}_k \otimes \mathrm{Tr}_n]P_{Wx}$, the minimum output entropy of the channel F is

$$H^{\min}(F) = \min_{x \in \mathbb{C}^d,\, \|x\|=1} H(F(P_x)) = \min_{y \in \mathrm{Im} W,\, \|y\|=1} H([\mathrm{id}_k \otimes \mathrm{Tr}_n]P_y) = H^{\min}(V),$$

where $V = \mathrm{Im} W \subset \mathbb{C}^k \otimes \mathbb{C}^n$ is a random subspace of dimension d.

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  where $V = \mathrm{Im}W \subset \mathbb{C}^k \otimes \mathbb{C}^n$ is a random subspace of dimension d.
- Computing $H^{\min}(F)$ can be thus reduced to finding K_V, where V is the image of the isometry defining F:

$$\boxed{H^{\min}(F) = \min_{\lambda\in K_V} H(\lambda).}$$

Main result

For the rest of the talk, we consider the following asymptotic regime: k fixed, $n \to \infty$, and $d \sim tkn$, for a fixed parameter $t \in (0, 1)$.

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Theorem (Belinschi, Collins, N. '10)

For a sequence of uniformly distributed random subspaces V_n, the set K_{V_n} of singular values of unit vectors from V_n converges (almost surely, in the Hausdorff distance) to a deterministic, convex subset $K_{k,t}$ of the probability simplex Δ_k

$$K_{k,t} := \{\lambda \in \Delta_k \mid \forall x \in \Delta_k, \langle \lambda, x \rangle \leq \|x\|_{(t)}\}.$$

Corollary: exact limit of the minimum output entropy

By the previous theorem, in the specific asymptotic regime t, k fixed, $n \to \infty$, $d \sim tkn$, we have the following a.s. convergence result for random quantum channels F (defined via random isometries $W : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n$):

$$\lim_{n\to\infty} H^{\min}(F) = \min_{\lambda \in K_{k,t}} H(\lambda).$$

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It is not just a bound, the exact limit value is obtained. However, the set $K_{k,t}$ is not explicit, and minimizing entropy functions is difficult.

Theorem (Belinschi, Collins, N. '13, in preparation)

The minimum entropy element of $K_{k,t}$ is of the form $(a, b, b, \ldots, b)$. The lowest dimension for which a violation of the additivity for $H^{\min}$ can be observed is $k = 183$. For large k, violations of size $1 - \varepsilon$ bits can be obtained.

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  - classical probability spaces $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$;
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- Example: if $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are free, then

$$\tau(a_1 b_1 a_2 b_2) = \tau(a_1 a_2)\tau(b_1)\tau(b_2) + \tau(a_1)\tau(a_2)\tau(b_1 b_2) - \tau(a_1)\tau(b_1)\tau(a_2)\tau(b_2).$$

Asymptotic freeness of random matrices

Theorem (Voiculescu '91)

Let (A_n) and (B_n) be sequences of $n \times n$ matrices such that A_n and B_n converge in distribution (with respect to $n^{-1}\mathrm{Tr}$) for $n \to \infty$. Furthermore, let (U_n) be a sequence of Haar unitary $n \times n$ random matrices. Then, A_n and $U_n B_n U_n^*$ are asymptotically free for $n \to \infty$.

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If A_n, B_n are matrices of size n, whose spectra converge towards μ_a, μ_b, the spectrum of $A_n + U_n B_n U_n^*$ converges to $\mu_a \boxplus \mu_b$; here, $\mu_a \boxplus \mu_b$ is the distribution of $a + b$, where $a, b \in (\mathcal{A}, \tau)$ are free random variables having distributions resp. μ_a, μ_b.

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If A_n, B_n are matrices of size n such that $A_n \geq 0$, whose spectra converge towards μ_a, μ_b, the spectrum of $A_n^{1/2} U_n B_n U_n^* A_n^{1/2}$ converges to $\mu_a \boxtimes \mu_b$.

Example: truncation of random matrices

Let $P_n \in \mathbb{M}_n$ a projection of rank $n/2$; its eigenvalues are 0 and 1, with multiplicity $n/2$. Hence, the distribution of P_n converges, when $n \to \infty$, to the Bernoulli probability measure $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

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Let $C_n \in \mathbb{M}_{n/2}$ be the top $n/2 \times n/2$ corner of $U_n P_n U_n^*$, with U_n a Haar random unitary matrix. What is the distribution of C_n ?

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Let $C_n \in \mathbb{M}_{n/2}$ be the top $n/2 \times n/2$ corner of $U_n P_n U_n^*$, with U_n a Haar random unitary matrix. What is the distribution of C_n ? Up to zero blocks, $C_n = Q_n(U_n P_n U_n^*)Q_n$, where Q_n is the diagonal orthogonal projection on the first $n/2$ coordinates of $\mathbb{C}^n$. The distribution of Q_n converges to $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

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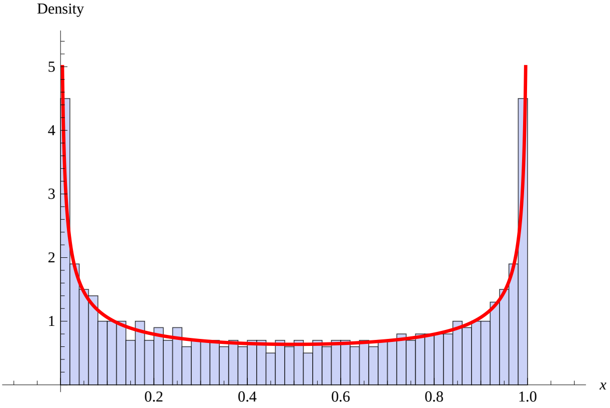
Free probability theory tells us that the distribution of C_n will converge to

$$(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1) \boxtimes (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1) = \frac{1}{\pi\sqrt{x(1-x)}}\mathbf{1}_{[0,1]}(x)dx,$$

which is the arcsine distribution.

Example: truncation of random matrices

Histogram of eigenvalues of a truncated randomly rotated projector of relative rank $1/2$ and size $n = 4000$; in red, the density of the arcsine distribution.



The t-norm

Definition

For a positive integer k, embed $\mathbb{R}^k$ as a self-adjoint real subalgebra $\mathcal{R}$ of a C^*-ncps $(\mathcal{A}, \tau)$, so that $\tau(x) = (x_1 + \cdots + x_k)/k$. Let p_t be a projection of rank $t \in (0, 1]$ in $\mathcal{A}$, free from $\mathcal{R}$. On the real vector space $\mathbb{R}^k$, we introduce the following norm, called the (t)-norm:

$$\|x\|_{(t)} := \|p_t x p_t\|_\infty,$$

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- One can show that $\|\cdot\|_{(t)}$ is indeed a norm, which is permutation invariant.
- When $t > 1 - 1/k$, $\|\cdot\|_{(t)} = \|\cdot\|_\infty$ on $\mathbb{R}^k$.
- $\lim_{t \to 0^+} \|x\|_{(t)} = k^{-1} |\sum_i x_i|$.

Corners of randomly rotated projections

Theorem (Collins '05)

In $\mathbb{C}^n$, choose at random according to the Haar measure two independent subspaces V_n and V'_n of respective dimensions $q_n \sim sn$ and $q'_n \sim tn$ where $s, t \in (0,1]$. Let P_n (resp. P'_n) be the orthogonal projection onto V_n (resp. V'_n). Then,

$$\lim_n \|P_n P'_n P_n\|_\infty = \varphi(s,t) = \sup \operatorname{supp}((1-s)\delta_0 + s\delta_1) \boxtimes ((1-t)\delta_0 + t\delta_1),$$

with

$$\varphi(s,t) = \begin{cases} s + t - 2st + 2\sqrt{st(1-s)(1-t)} & \text{if } s+t<1; \\ 1 & \text{if } s+t \geq 1. \end{cases}$$

Hence, we can compute

$$\| \underbrace{1, \cdots, 1}_{j \text{ times}}, \underbrace{0, \cdots, 0}_{k-j \text{ times}} \|_{(t)} = \varphi(\frac{j}{k}, t).$$

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\begin{aligned}
\max_{x \in V, \|x\|=1} \lambda_1(x) &= \max_{x \in V, \|x\|=1} \lambda_1([\mathrm{id}_k \otimes \mathrm{Tr}_n] P_x) \\
&= \max_{x \in V, \|x\|=1} \|[\mathrm{id}_k \otimes \mathrm{Tr}_n] P_x\| \\
&= \max_{x \in V, \|x\|=1} \max_{y \in \mathbb{C}^k, \|y\|=1} \mathrm{Tr}\left[([\mathrm{id}_k \otimes \mathrm{Tr}_n] P_x) \cdot P_y\right] \\
&= \max_{x \in V, \|x\|=1} \max_{y \in \mathbb{C}^k, \|y\|=1} \mathrm{Tr}\left[P_x \cdot P_y \otimes \mathrm{I}_n\right] \\
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Limit of $\|P_V \cdot P_y \otimes \mathrm{I}_n\|_\infty$ for fixed y and random V ?

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- We can take the max over y at no cost, by considering a finite net of y's, since k is fixed.
- To get the full result $\limsup_{n\to\infty} K_{V_n} \subset K_{k,t}$, use $\langle \lambda, x\rangle$ (for all directions x) instead of λ_1.

Mulțumesc !

Collins, N. - *Random quantum channels II: Entanglement of random subspaces, Rényi entropy estimates and additivity problems*.

Belinschi, Collins, N. - *Laws of large numbers for eigenvectors and eigenvalues associated to random subspaces in a tensor product*.

Belinschi, Collins, N. - *1 bit violations for the additivity of the minimum ouptut entropy of quantum channels* (in preparation).