Entanglement of random subspaces

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joint work with
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Autrans, July 16th 2013

Outline of the talk

1. Additivity problems in QIT
   - Quantum states and channels
   - The additivity problem
   - From channels to subspaces
2. Entanglement of subspaces
   - Singular values of matrices / bipartite vectors
   - Minimal entanglement vs. MOE
   - The set K_V
3. Random subspaces
   - Statement of the main result and applications
   - Free probability - a review
   - Sketch of the proof

States and channels in quantum information theory

- Quantum states with n degrees of freedom are described by density matrices

$$\rho \in \mathbb{M}_n^{1,+} = \mathrm{End}^{1,+}(\mathbb{C}^n); \qquad \mathrm{Tr}\rho = 1 \text{ and } \rho \geq 0.$$

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  - Trace preservation TP: $\mathrm{Tr}[F(X)] = \mathrm{Tr}(X)$ for all X.
- Examples: $F_U(X) = UXU^*$, $F_{dep}(X) = \mathrm{Tr}(X)I_k/k$.

Some notions of entropy

- Let $\Delta_k = \{\lambda \in \mathbb{R}^k \,:\, \lambda_i \geq 0, \, \sum_i \lambda_i = 1, \}$ be the probability simplex. We write $\Delta_k^{\downarrow}$ for the set of ordered probability vectors, $\lambda_1 \geq \cdots \geq \lambda_k$.
- The Shannon entropy of a probability vector $\lambda \in \Delta_k$

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- The entropy is additive: $H_p(X_1 \otimes X_2) = H_p(X_1) + H_p(X_2)$.

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- Additivity of $H_{p=1}^{\min}$ implies a simple formula for the capacity of channels to transmit classical information; in particular, it implies the additivity of the classical capacity. Moreover, it is equivalent to the additivity of the Holevo capacity and the additivity of the entanglement of formation

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- But... the Additivity Conjecture is false ! [Hayden, Winter '08 for $p > 1$, Hastings '09 for $p = 1$]
- Counterexamples: mostly random channels. Deterministic counterexamples: '02 Werner & Holevo ($p > 4.79$), '07 Cubitt et al ($p < 0.11$) and '09 Grudka et al ($p > 2$).

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Theorem (Stinespring dilation)

For any channel $F : \mathbb{M}_d \to \mathbb{M}_k$ *there exists an isometry* $W : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n$ *such that*

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$$H^{\min}(F) = \min_{x \in \mathbb{C}^d,\, \|x\|=1} H(F(P_x)) = \min_{y \in \mathrm{Im} W,\, \|y\|=1} H([\mathrm{id}_k \otimes \mathrm{Tr}_n]P_y),$$

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- The MOE $H^{\min}(F)$ depends only on the subspace V.

Eigen- and singular values

Singular value decomposition of a matrix $X \in \mathbb{M}_{k\times n}(\mathbb{C})$ $(k \leq n)$

$$X = U\Sigma V^* = \sum_{i=1}^{k} \sqrt{\lambda_i(XX^*)}e_i f_i^*,$$

where e_i, f_i are orthonormal families in $\mathbb{C}^k$, $\mathbb{C}^n$, and $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$ are the (squares of the) singular values of X, or the eigenvalues of XX^*.

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The numbers $\lambda_i(x)$ are also eigenvalues of the reduced density matrix

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Recall that we are interested in computing

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Example: $V_{ent} = \{x : \forall r, \sum_{i+j=r} x_{ij} = 0\}$.

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- Recovering minimum entropies:

$$H_p^{\min}(F) = H_p^{\min}(V) = \min_{\lambda \in K_V} H_p(\lambda).$$

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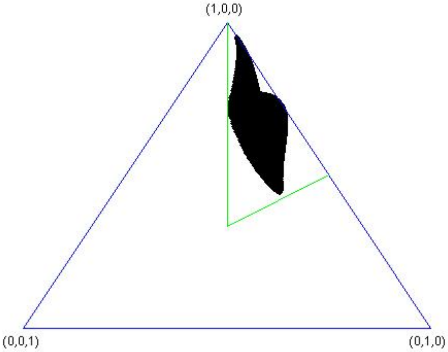
- Fact: singular values of vectors in V come in pairs.
- Hence, the least entropy vector in V is as above, with $e \perp f$ and $H(x) = \log 2$.
- Thus, $H^{\min}(V) = \log 2$ and one can show that

$$K_V = \{(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots) \in \Delta_k \ : \ \lambda_i \geq 0, \sum_i \lambda_i = 1/2\}.$$

Examples

$V = \mathrm{span}\{G_1, G_2\}$, where $G_{1,2}$ are 3×3 independent Ginibre random matrices.



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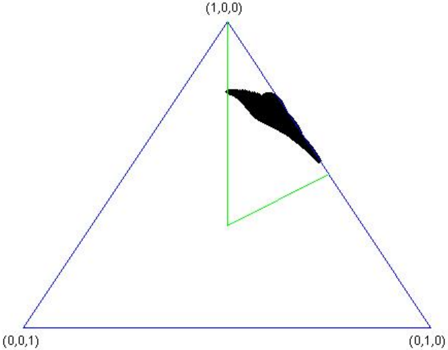

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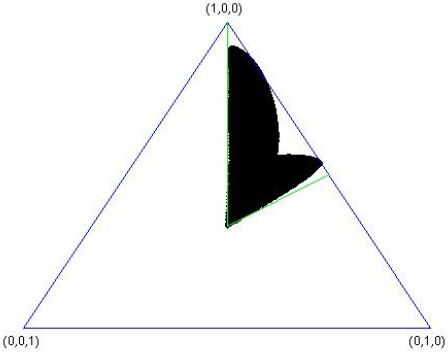

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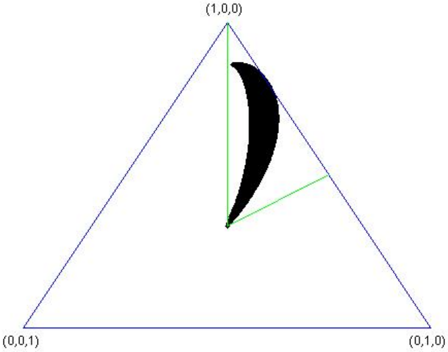

A big open problem

Find explicit examples of subspaces V with

1. large $\dim V$;
2. large $H^{\min}(V)$.

Random subspaces

We are interested in random subspaces (or random channels).

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- There is an uniform (or Haar) measure on the set of isometries $\{W : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n \, : \, WW^* = \mathrm{I}_d\}$: take a $kn \times kn$ Haar distributed random unitary matrix $U \in \mathcal{U}(kn)$ and take W to be the restriction of U to the first d coordinates.

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- We call random quantum channels the probability distribution obtained as the push-forward of this measure through the Stinespring dilation.
- A random subspace is the image of a random isometry, $V = \mathrm{Im} W$.
- Equivalently, $V = \mathrm{span}\{U_1, \ldots, U_d\}$, where U_i are the columns of a Haar random unitary matrix $U \in \mathcal{U}(kn)$.

Main result

For the rest of the talk, we consider the following asymptotic regime: k fixed, $n \to \infty$, and $d \sim tkn$, for a fixed parameter $t \in (0, 1)$.

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Theorem (Belinschi, Collins, N. '10)

For a sequence of uniformly distributed random subspaces V_n, the set K_{V_n} of singular values of unit vectors from V_n converges (almost surely, in the Hausdorff distance) to a deterministic, convex subset $K_{k,t}$ of the probability simplex Δ_k

$$\boxed{K_{k,t} := \{\lambda \in \Delta_k \mid \forall x \in \Delta_k, \langle \lambda, x \rangle \leq \|x\|_{(t)}\}.}$$

Corollary: exact limit of the minimum output entropy

By the previous theorem, in the specific asymptotic regime t, k fixed, $n \to \infty$, $d \sim tkn$, we have the following a.s. convergence result for random quantum channels F (defined via random isometries $W : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n$):

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Theorem (Belinschi, Collins, N. '13)

The minimum entropy element of $K_{k,t}$ is of the form $(a, b, b, \ldots, b)$. The lowest dimension for which a violation of the additivity for $H^{\min}$ can be observed is $k = 183$. For large k, violations of size $1 - \varepsilon$ bits can be obtained.

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  - classical probability spaces $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$;
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- Example: if $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are free, then

$$\tau(a_1 b_1 a_2 b_2) = \tau(a_1 a_2)\tau(b_1)\tau(b_2) + \tau(a_1)\tau(a_2)\tau(b_1 b_2) - \tau(a_1)\tau(b_1)\tau(a_2)\tau(b_2).$$

Asymptotic freeness of random matrices

Theorem (Voiculescu '91)

Let (A_n) and (B_n) be sequences of $n \times n$ matrices such that A_n and B_n converge in distribution (with respect to $n^{-1}\mathrm{Tr}$) for $n \to \infty$. Furthermore, let (U_n) be a sequence of Haar unitary $n \times n$ random matrices. Then, A_n and $U_n B_n U_n^*$ are asymptotically free for $n \to \infty$.

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If A_n, B_n are matrices of size n, whose spectra converge towards μ_a, μ_b, the spectrum of $A_n + U_n B_n U_n^*$ converges to $\mu_a \boxplus \mu_b$; here, $\mu_a \boxplus \mu_b$ is the distribution of $a + b$, where $a, b \in (\mathcal{A}, \tau)$ are free random variables having distributions resp. μ_a, μ_b.

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If A_n, B_n are matrices of size n such that $A_n \geq 0$, whose spectra converge towards μ_a, μ_b, the spectrum of $A_n^{1/2} U_n B_n U_n^* A_n^{1/2}$ converges to $\mu_a \boxtimes \mu_b$.

Example: truncation of random matrices

Let $P_n \in \mathbb{M}_n$ a projection of rank $n/2$; its eigenvalues are 0 and 1, with multiplicity $n/2$. Hence, the distribution of P_n converges, when $n \to \infty$, to the Bernoulli probability measure $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

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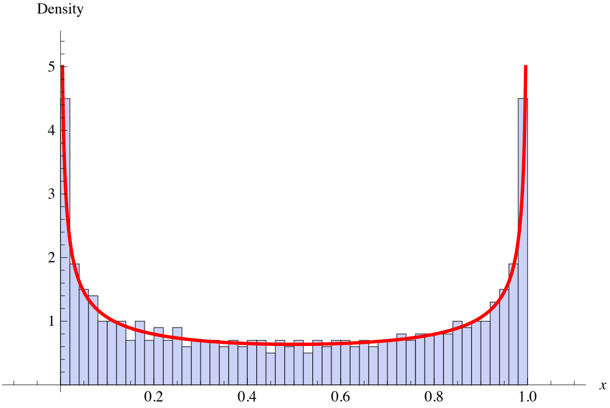
Free probability theory tells us that the distribution of C_n will converge to

$$(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1) \boxtimes (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1) = \frac{1}{\pi\sqrt{x(1-x)}}\mathbf{1}_{[0,1]}(x)dx,$$

which is the arcsine distribution.

Example: truncation of random matrices

Histogram of eigenvalues of a truncated randomly rotated projector of relative rank $1/2$ and size $n = 4000$; in red, the density of the arcsine distribution.



The t-norm

Definition

For a positive integer k, embed $\mathbb{R}^k$ as a self-adjoint real subalgebra $\mathcal{R}$ of a C^*-ncps $(\mathcal{A}, \tau)$, so that $\tau(x) = (x_1 + \cdots + x_k)/k$. Let p_t be a projection of rank $t \in (0, 1]$ in $\mathcal{A}$, free from $\mathcal{R}$. On the real vector space $\mathbb{R}^k$, we introduce the following norm, called the (t)-norm:

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- One can show that $\|\cdot\|_{(t)}$ is indeed a norm, which is permutation invariant.
- When $t > 1 - 1/k$, $\|\cdot\|_{(t)} = \|\cdot\|_\infty$ on $\mathbb{R}^k$.
- $\lim_{t \to 0^+} \|x\|_{(t)} = k^{-1} |\sum_i x_i|$.

Corners of randomly rotated projections

Theorem (Collins '05)

In $\mathbb{C}^n$, choose at random according to the Haar measure two independent subspaces V_n and V'_n of respective dimensions $q_n \sim sn$ and $q'_n \sim tn$ where $s, t \in (0,1]$. Let P_n (resp. P'_n) be the orthogonal projection onto V_n (resp. V'_n). Then,

$$\lim_n \|P_n P'_n P_n\|_\infty = \varphi(s,t) = \sup \operatorname{supp}((1-s)\delta_0 + s\delta_1) \boxtimes ((1-t)\delta_0 + t\delta_1),$$

with

$$\varphi(s,t) = \begin{cases} s + t - 2st + 2\sqrt{st(1-s)(1-t)} & \text{if } s+t < 1; \\ 1 & \text{if } s+t \geq 1. \end{cases}$$

Hence, we can compute

$$\|\underbrace{1,\cdots,1}_{j \text{ times}},\underbrace{0,\cdots,0}_{k-j \text{ times}}\|_{(t)} = \varphi(\frac{j}{k}, t).$$

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\max_{x \in V, \|x\|=1} \lambda_1(x) &= \max_{x \in V, \|x\|=1} \lambda_1([\mathrm{id}_k \otimes \mathrm{Tr}_n] P_x) \\
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&= \max_{x \in V, \|x\|=1} \max_{y \in \mathbb{C}^k, \|y\|=1} \mathrm{Tr}\left[([\mathrm{id}_k \otimes \mathrm{Tr}_n] P_x) \cdot P_y\right] \\
&= \max_{x \in V, \|x\|=1} \max_{y \in \mathbb{C}^k, \|y\|=1} \mathrm{Tr}\left[P_x \cdot P_y \otimes \mathrm{I}_n\right] \\
&= \max_{y \in \mathbb{C}^k, \|y\|=1} \max_{x \in V, \|x\|=1} \mathrm{Tr}\left[P_x \cdot P_y \otimes \mathrm{I}_n\right] \\
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Limit of $\|P_V \cdot P_y \otimes \mathrm{I}_n\|_\infty$ for fixed y and random V ?

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- For fixed y, P_V and $P_y \otimes \mathrm{I}_n$ are independent projectors of relative ranks t and $1/k$ respectively.

The set $K_{k,t}$ and t-norms

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- To get the full result $\limsup_{n \to \infty} K_{V_n} \subset K_{k,t}$, use $\langle \lambda, x \rangle$ (for all directions x) instead of λ_1.

Thank you!

Collins, N. - *Random quantum channels II: Entanglement of random subspaces, Rényi entropy estimates and additivity problems*.

Belinschi, Collins, N. - *Laws of large numbers for eigenvectors and eigenvalues associated to random subspaces in a tensor product*.

Belinschi, Collins, N. - *Almost one bit violation for the additivity of the minimum output entropy*.