3-4/09/13 | Random matrices, with a view towards free probability and QIT

1

Plan: prove 2 theorems and introduce main objects in free prob. + give applications to QIT

Def [Wishart ensemble]

- $X \in M_{d \times s}(\mathbb{C})$ with $\{X_{ij}\}$ iid. complex Gaussian,

$$\mathbb{E} X_{ij} = 0 \qquad \mathbb{E}|X_{ij}|^2 = 1.$$

(Ginibre ensemble)

- $W \in M_d(\mathbb{C})$ $\quad W = XX^*$ is called a Wishart random matrix of parameters (d,s)

Notation For $A \in M_d^{sa}(\mathbb{C})$, the empirical eigenvalue distribution

$$\mu_A = \frac{1}{d} \sum_{i=1}^{d} \delta_{\lambda_i(A)} \quad \leftrightarrow \quad \text{histogram of ev of } A.$$

Thm 1 [Marchenko, Pastur '67]

Let $W_d \sim$ Wishart (d, s_d) such that $s_d \sim cd$ for $c>0$ cst.

Then, a.s. as $d \to \infty$ $\quad \mu_{\frac{W_d}{d}} \xrightarrow{(dist)} \pi_c$.

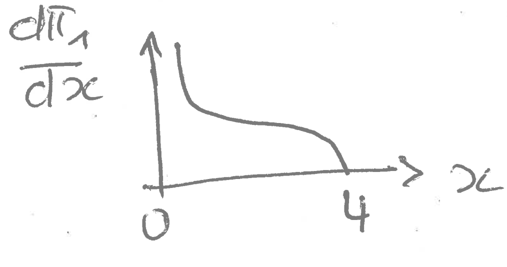
L2

where Π_c is the M-P dist of param. c

$$\Pi_c = \max(1-c, 0)\,\delta_0 + \frac{\sqrt{(b-x)(x-a)}}{2\pi x}\,\mathbb{1}_{[a,b]}(x)\,dx,$$

where $a := (\sqrt{c}-1)^2$, $b := (\sqrt{c}+1)^2$

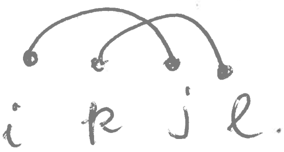
Ex $c = 1$



Def [non-crossing partitions]

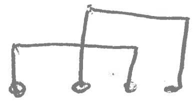
α partition of $\{1, \ldots, p\} =: [p]$ is NC iff.

$\nexists\ i<k<j<l$ st. $i \overset{\alpha}{\sim} j$ $k \overset{\alpha}{\sim} l$.



Ex $p = 4$

, but not



$\{1,3\},\{2,4\}$

for $\alpha \in NC(p)$, note $\#\alpha$ = # blocks of α.

Remark We have

$$\int x^p \, d\Pi_c(x) = \sum_{\alpha \in NC(p)} c^{\#\alpha}$$

In particular $(c=1)$ $\int x^p \, d\Pi_1(x) = \# NC(p) = Cat_p = \frac{1}{p+1}\binom{2p}{p}$

Partial transposition

$$W \in M_d \otimes M_n \simeq M_{dn}(\mathbb{C})$$

$$W^\Gamma := (id \otimes t)(W) \qquad W^\Gamma_{ia,jb} = W_{ib,ja}$$

t ← transposition

Thm 2 [Banica, N. '12]

$W \in M_d \otimes M_n(\mathbb{C}) \sim \text{Wishart}(dn, dm)$

Then, (m, n fixed)

$$\mu_{\frac{W_d^\Gamma}{d}} \xrightarrow[d \to \infty]{(\text{dist})} \pi_{\frac{m(n+1)}{2}} \boxplus \pi^{(-)}_{\frac{m(n-1)}{2}}$$

Lecture 2: what is "$\boxplus$" → free additive conv.

$X \sim \mu \quad (-X) \sim \mu^{(-)}$ ← flip the measure

Proof of Thm 1

Steps:

1) Moment formula for W_d:

$$\forall d, s, p \quad \mathbb{E}\,\mathrm{Tr}\, W^p = \sum_{\alpha \in S_p} d^{\#(\alpha^{-1}\gamma)} s^{\#\alpha}$$

where $\#\alpha = \#$ cycles of α

$\gamma \in S_p$ full cycle

$\gamma = (p \ (p-1) \ \cdots \ 3 \ 2 \ 1)$

2) Convergence in moments

$$\forall p \geq 1 \quad \lim_{d \to \infty} \frac{1}{d} \mathbb{E} \operatorname{Tr} \left(\frac{W_d}{d} \right)^p = \sum_{\alpha \in NC(p)} c^{\#\alpha}$$

$$\boxed{\begin{array}{l} \text{! at fixed } d : \sum_{\alpha \in S_p} \\ \circ \text{ at the limit } d \to \infty : \sum_{\alpha \in \underbrace{NC(p)}} \end{array}}$$

↳ signature of free probability.

3) Stieltjes inversion : going from a moment formula for π_c to the density.

4) Almost sure convergence : use Borel-Cantelli

$$\sum_{d=1}^{\infty} \mathbb{E}\left[(m_p - \mathbb{E} m_p)^2 \right] < \infty$$

$$m_p = \frac{1}{d} \mathbb{E} \operatorname{Tr} \left(\frac{W_d}{d} \right)^p.$$

Will do:

- tool for step 1)
- step 2)

Step 1 want $\mathbb{E}\,\mathrm{Tr}\,W^p$ → expand.

$$\mathbb{E}\,\mathrm{Tr}\,W_p = \sum_{\substack{1\le i_1\cdots i_p\le d\\ 1\le j_1\cdots j_p\le s}} \mathbb{E}\, X_{i_1j_1}\overline{X}_{i_2j_1}X_{i_2j_2}\overline{X}_{i_3j_2}\cdots X_{i_pj_p}\overline{X}_{i_1j_p}$$

and use

Wick's lemma $X\in\mathbb{R}^{2p}$ Gaussian vector, then

$$\mathbb{E}[X_1\cdots X_{2p}] = \sum_{\substack{\{i_1j_1\},\dots\{i_pj_p\}\\ \text{pairing of } [2p]}} \prod_{k=1}^{p} \mathbb{E}[X_{i_k}X_{j_k}]$$

Complex Gaussian version: only pairings between $X \leftrightarrow \overline{X}$ are allowed.

In our case
$$\mathbb{E}[X_{ij}\overline{X}_{i'j'}] = \begin{cases} 0 & \text{if indep } (i,j)\neq(i'j') \\ 1 & \text{if identical } (i,j)=(i'j') \end{cases}$$

⇝ combinatorial counting of pairings $\leftrightarrow S_p$.

Step 2 Asymptotics. $d \to \infty$

$s \to \infty$ $s \sim cd$.

$c > 0$ fixed.

$$\mathbb{E}\,\mathrm{Tr}\, W^p \sim \sum_{\alpha \in S_p} d^{\#(\alpha^{-1}\gamma) + \#\alpha} c^{\#\alpha}.$$

dominating terms: $\boxed{\max_{\alpha \in S_p} \#(\alpha^{-1}\gamma) + \#\alpha}$

(γ = full cycle
$\gamma = (p\ (p-1) \cdots 3\ 2\ 1)$)

Notation: length of a permutation $\alpha \in S_p$

$|\alpha| = \min\{k : \exists\, \tau_1 \cdots \tau_k$ transpositions such that $\alpha = \tau_1 \cdots \tau_k\}$

Ex
$|id| = 0$; $|\gamma| = p - 1$

Proposition [Biane]

- $|\alpha| + \#\alpha = p$.
- $d(\alpha, \beta) = |\alpha^{-1}\beta|$ is a distance on S_p.
  $\hookrightarrow$ in particular, $|\alpha| + |\alpha^{-1}\beta| \geq |\beta|$.
- $|\alpha| + |\alpha^{-1}\beta| = |\beta|$ iff
  $\to \forall\, C_\alpha$ cycle of α, $\exists\, C_\beta$ cycle of β st $C_\alpha \subset C_\beta$

→ $\forall c_\beta$ cycle of β,

→ α defines a NC part. on c_β.

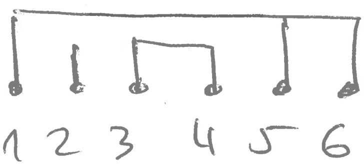
→ α respects the ordering of c_β

(we write $id \to \alpha \to \beta$ geodesic)

Ex $\beta = \gamma$. $id \to \alpha \to \gamma$ geodesic iff

- cycles of α are NC
- ________

$\alpha = (651)(2)(43)$



Back to our optimization problem:

$$\#(\alpha^{-1}\gamma) + \#\alpha = 2p - (\underbrace{|\alpha| + |\alpha^{-1}\gamma|}_{\geq |\gamma| = p-1}) \leq p+1$$

with equality iff $id \to \alpha \to \gamma$ geodesic.

<u>But</u>

$$\boxed{! \; \{\alpha \in S_p : id \to \alpha \to \gamma\} \leftrightarrow NC(p)}$$

(just order the blocks of a NC partition!)

So.

$$\mathbb{E}\,\mathrm{Tr}\,W_d^p \sim d^{p+1} \sum_{\alpha \in NC(p)} c^{\#\alpha}$$, done! $\square$

2nd lecture

1st lecture:

$W \sim \mathrm{Wishart}(d,s)$ if $W = XX^*$ with

$X \in M_{d\times s}(\mathbb{C})$ X_{ij} iid $\mathbb{C}$-Gaussians

Thm 1 $W_d \sim \mathrm{Wishart}(d, cd)$

$$\frac{1}{d}\mathbb{E}\,\mathrm{Tr}\left(\frac{W_d}{d}\right)^p \xrightarrow[d\to\infty]{} \sum_{\alpha \in NC(p)} c^{\#\alpha}$$

(M-P dist.)

Thm 2 $W_d \sim \mathrm{Wishart}(dn, dm)$

$W_d^\Gamma = (\mathrm{id} \otimes t)\, W_d.$

$$\frac{1}{dn}\mathbb{E}\,\mathrm{Tr}\left(\frac{W_d^\Gamma}{d}\right)^p \xrightarrow[d\to\infty]{} \int x^p \, d\tilde{\pi}_{m,n}(x)$$

$$\tilde{\pi}_{m,n} : \pi_{\frac{m(n+1)}{2}} \boxplus \pi^{(-)}_{\frac{m(n-1)}{2}}$$

Proof of Thm 2

1) Moment formula:

effect of the transposition

$$\mathbb{E}\,\mathrm{Tr}(W_d^\Gamma)^p = \sum_{\alpha\in S_p} d^{\#(\alpha^{-1}\gamma)}\, n^{\#(\alpha^{-1}\gamma^{-1})}\,(dm)^{\#\alpha}$$

$$\longrightarrow \quad d^{\#(\alpha^{-1}\gamma)}\, d^{\#\alpha} \Rightarrow \text{Same dominant terms as in Thm 1}$$

2) Asymptotics.

$$\frac{1}{dn}\,\mathbb{E}\,\mathrm{Tr}\left(\frac{W^\Gamma}{d}\right)^p \longrightarrow \sum_{\alpha\in NC(p)} m^{\#\alpha}\, n^{\#(\alpha^{-1}\gamma^{-1})-1}$$

Lemma $\alpha\in NC(p)\subset S_p$, then

$\#(\alpha^{-1}\gamma^{-1}) = \#(\alpha\gamma) = 1 + e(\alpha)$

where $e(\alpha) = \#$ even blocks/cycles of α.

$$\xrightarrow[d\to\infty]{} \sum_{\alpha\in NC(p)} m^{\#\alpha}\, n^{\# e(\alpha)}$$

Next make sense of the moment

"identify the" measure

$\leadsto$ free probability.

Intro to free prob. theory

- D. Voiculescu '80s
- wanted to solve pbs in op. alg.
- connection to RMT discovered later. 90s.
- ⋮

Def [non-commutative prob. space ncps]

A ncps is an algebra $\mathcal{A}$, $1 \in \mathcal{A}$ and a lin. f. $\varphi : \mathcal{A} \to \mathbb{C}$ s.t. $\varphi(1) = 1$

⇝ more analytic structure: C^*-alg, vN alg..

Examples

1) classical: $\left(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E} \right)$

1') $L^{\infty -} = \bigcap_{p \geq 1} L^p$: all moments exist (Gaussian)

2) matrices $\left(M_d(\mathbb{C}), \frac{1}{d} \mathrm{Tr} \right)$

3) random matrices

$$\left(M_d(L^{\infty -}), \frac{1}{d} \mathbb{E}\,\mathrm{Tr} \right)$$

4) group algebras. G group

$$\mathbb{C}G = \left\{ \sum_{g \in G}^{\text{finite}} x_g g \, , \, x_g \in \mathbb{C} \right\}$$

$$\tau\left(\sum x_g g\right) = x_e .$$

→ we have random variables (which might not commute)

→ need a notion of non-comm. independence

Def [free indep. / freeness]

Sub-alg. $\{\mathcal{A}_i\}_{i \in I}$ $1 \in \mathcal{A}_i \ \forall i$ are free iff

$$\left.\begin{array}{l} \varphi(a_j) = 0 \quad \forall j \\ a_j \in \mathcal{A}_{i(j)} \ \forall j \\ i(1) \neq i(2) \, ; \, i(2) \neq i(3) \, ; \\ \quad \cdots \, i(k-1) \neq i(k) \end{array}\right\} \Rightarrow \varphi(a_1 \cdots a_k) = 0$$

Example • a, b free. what is $\varphi(ab) = ?$

$$\varphi(ab) \underset{\substack{\uparrow \\ \text{linearity} \\ \text{of } \varphi}}{=} \varphi(a)\varphi(b) + \underbrace{\varphi\Big[\underbrace{(a - \varphi(a) \cdot 1) \cdot (b - \varphi(b) 1)}_{\text{centered + freeness}}\Big]}_{=0.}$$

So $\quad \varphi(ab) = \varphi(a)\varphi(b)$ if a,b free

! In general, freeness allows us to compute joint moments in terms of the marginals.

Ex: $\{a_1, a_2\}$ free from $\{b_1, b_2\}$. Then

$$\varphi(a_1 b_1 a_2 b_2) = \varphi(a_1 a_2)\varphi(b_1)\varphi(b_2) +$$
$$+ \varphi(a_1)\varphi(a_2)\varphi(b_1 b_2) -$$
$$- \varphi(a_1)\varphi(a_2)\varphi(b_1)\varphi(b_2)$$

$\rightsquigarrow$ more complicated formulas than in the classical, commutative case

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free cumulants

a,b free. what is $\varphi((a+b)^p)$?

Classical prob: use Fourier transform!

$$\log \mathcal{F}_{\mu * \nu} = \log \mathcal{F}_\mu + \log \mathcal{F}_\nu$$

Free prob: R-transform. Its series coeffs: free cumulants

$\rightsquigarrow$ linearize free sum

$\varphi : \mathcal{A} \to \mathbb{C}$

<u>Notation</u>

- $\varphi_n : \mathcal{A}^p \to \mathbb{C}$

$$(a_1, \ldots, a_p) \mapsto \varphi(a_1 \cdots a_p)$$

- $\alpha \in NC(p)$

$$\varphi_\alpha(a_1, \ldots, a_p) := \prod_{b \in \alpha} \varphi_{|b|}(a_{b_1}, \ldots, a_{b_{|b|}})$$

(b block of α)

Ex

$$\varphi_{\sqcap\!\sqcap\!\sqcap}(a, b, c, d) = \varphi(abcd)$$

$$\varphi_{||||}(a, b, c, d) = \varphi(a)\varphi(b)\varphi(c)\varphi(d)$$

$$\varphi_{\sqcap|\sqcap\!\sqcap}(a, b, c, d) = \varphi(acd)\,\varphi(b)$$

<u>Def / Prop</u> $\exists!$ family of multi-lin functionals $k_n : \mathcal{A}^n \to \mathbb{C}$ s.t

$$\forall p \quad \varphi_p(a_1, \ldots, a_p) = \sum_{\alpha \in NC(p)} k_\alpha(a_1, \ldots, a_p)$$

[moment-cumulant formula]

<u>Ex</u>

$$\varphi_1(a) = k_{|}(a) = k_1(a)$$

$$\varphi_2(a, b) = k_{\sqcap}(a, b) + k_{||}(a, b)$$

$$= k_2(a, b) + k_1(a)\,k_1(b)$$

$\Rightarrow k_2(a,b) = \varphi(ab) - \varphi(a)\varphi(b) = \mathrm{Cov}(a,b)$ 14

$\to$ one can algorithmically compute the free cumulants from the mom-cum formula

Sums of free random variables

Thm a,b free $\Rightarrow \forall p.\ k_p(\underbrace{a+b,\_,a+b}_{p \text{ times}}) = k_p(\underbrace{a,\_\,a}_{p})$

$+ k_p(\underbrace{b,\_\_,b}_{p})$

Notation

if a has dist μ

b —— ν

the dist of $a+b$ is denoted $\mu \boxplus \nu$

Recall: $a \sim \pi_c$ (Thm 1)

$$\varphi(a^p) = \sum_{\alpha \in NC(p)} c^{\#\alpha} = \sum_{\alpha \in NC(p)} \prod_{b \in \alpha} c$$

So, $\forall r,\ k_r(a) = c$

The free cumulants of π_c dist are all c

Back to the proof of Thm 2

$$\varphi(\tilde{\pi}^{p}_{m,n}) = \sum_{\alpha \in NC(p)} m^{\#\alpha} n^{e(\alpha)}$$

$$= \sum_{\alpha \in NC(p)} \prod_{b \in \alpha} m\, n^{\mathbb{1}(|b| \text{ even})}$$

$\underline{\text{So}}$: $$k_r(\tilde{\pi}_{m,n}) = \begin{cases} mn, & r \text{ even} \\ m, & r \text{ odd} \end{cases}$$

$$= \frac{m(n+1)}{2} + (-1)^r \frac{m(n-1)}{2}$$

$$\Rightarrow \tilde{\pi}_{m,n} = \pi_{\frac{m(n+1)}{2}} \boxplus \pi^{(-)}_{\frac{m(n-1)}{2}}$$

$\square$