Entanglement of generic quantum states

Ion Nechita

Technische Universität München
and
CNRS, LPT Toulouse

joint work with Teodor Banica (Cergy),
Maria Jivulescu and Nicoale Lupa (Timisoara)

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Talk outline

1. Entanglement in QIT
2. Random quantum states
3. Thresholds for entanglement criteria
4. Random matrices and free probability

Quantum states

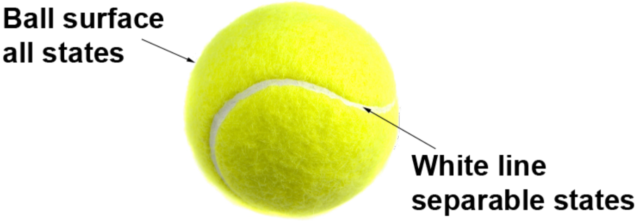
- Closed quantum systems with d degrees of freedom are described by pure states
$$|\psi\rangle \in \mathbb{C}^d, \qquad \|\psi\| = 1.$$
- Two quantum systems (Alice and Bob): $|\psi\rangle_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$.
- A state $|\psi\rangle_{AB}$ is called separable or product if it can be written as a tensor product
$$|\psi\rangle_{AB} = |x\rangle_A \otimes |y\rangle_B,$$
where $|x\rangle_A \in \mathbb{C}^{d_A}$ and $|y\rangle_B \in \mathbb{C}^{d_B}$.
- Non-separable states are called entangled.
- Examples with qubits ($d_A = d_B = 2$), $\mathbb{C}^2 = \mathrm{span}\{|0\rangle, |1\rangle\}$:
  - Separable: $|0\rangle_A \otimes |0\rangle_B$, $(|0\rangle_A + |1\rangle_A) \otimes |0\rangle_B/\sqrt{2}$;
  - Entangled: the Bell state $(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)/\sqrt{2}$

Pure entanglement

- We identify pure quantum states up to phases: for $\theta \in \mathbb{R}$,

$$|\psi\rangle = |e^{i\theta}\psi\rangle.$$

- Actually, quantum states live in a projective space $\mathbb{CP}^{d-1}$. As projective varieties, all bi-partite quantum states have dimension $d_A d_B - 1$, whereas product states have dimension $d_A + d_B - 2$, which is strictly smaller $\implies$ a generic pure state is entangled!



Pure state entanglement is easy

- For pure quantum states, entanglement can be detected and measured.
- The standard measure of the entanglement of a pure state $x = |x\rangle_{AB}$ is the entropy of entanglement

$$E(x) = -\sum_i s_i(x) \log s_i(x),$$

where $s_i(x)$ are the Schmidt coefficients of x:

$$|x\rangle = \sum_i \sqrt{s_i(x)}\, |e_i\rangle_A \otimes |f_i\rangle_B.$$

- For product states, $s(e \otimes f) = (1, 0, \ldots, 0)$ and thus $E(e \otimes f) = 0$. In general, $E(x) = 0 \iff x$ is product.
- For a Bell state $|\psi\rangle_{AB} = (|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)/\sqrt{2}$, one has $s(\psi) = (1/2, 1/2)$ and thus $E(\psi) = \log 2$.
- In general, $E(x) \in [0, \log \min(d_A, d_B)]$.

Mixed states and entanglement

- Open quantum systems with d degrees of freedom are described by density matrices or mixed states
$$\rho \in \mathcal{M}^{1,+}(\mathbb{C}^d); \qquad \mathrm{Tr}\rho = 1 \text{ and } \rho \geqslant 0.$$
- Pure states are the particular case of rank one projectors:
$$|\psi\rangle\langle\psi| \in \mathcal{M}^{1,+}(\mathbb{C}^d).$$
  They are the extreme points of the convex body of density matrices.
- Two quantum systems: $\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$.
- A mixed state ρ_{AB} is called separable if it can be written as a convex combination of product states
$$\rho_{AB} \in \mathcal{SEP} \iff \rho_{AB} = \sum_i t(i) \cdot \rho_A(i) \otimes \rho_B(i),$$
  with $t(i) \geqslant 0$, $\sum_i t(i) = 1$, $\rho_{A,B}(i) \in \mathcal{M}^{1,+}(\mathbb{C}^{d_{A,B}})$.
- Non-separable states are called entangled.

Mixed state entanglement is hard, but...

- Consider now a density matrix $\rho_{AB} \in \mathcal{M}_{d_A d_B}(\mathbb{C})$.
- Deciding if a given ρ_{AB} is separable is NP-hard [Gurvitz].
- Detecting entanglement for general states is a difficult, central problem in QIT.
- A map $f : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^d)$ is called
  - positive if $A \geqslant 0 \implies f(A) \geqslant 0$;
  - completely positive if $\mathrm{id}_k \otimes f$ is positive for all $k \geqslant 1$.
- If $f : \mathcal{M}(\mathbb{C}^{d_B}) \to \mathcal{M}(\mathbb{C}^{d_B})$ is CP, then for every state ρ_{AB} one has $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geqslant 0$.
- If $f : \mathcal{M}(\mathbb{C}^{d_B}) \to \mathcal{M}(\mathbb{C}^{d_B})$ is only positive, then for every separable state ρ_{AB}, one has $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geqslant 0$.
- Indeed,

$$[\mathrm{id}_{d_A} \otimes f] \left(\sum_i t(i) \cdot \rho_A(i) \otimes \rho_B(i) \right) = \sum_i t(i) \cdot \rho_A(i) \otimes f(\rho_B(i)) \geqslant 0,$$

since each term is positive semidefinite.

Entanglement detection via positive, but not CP maps

- Positive, but not CP maps f yield entanglement criteria: given ρ_{AB}, if $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \not\geqslant 0$, then ρ_{AB} is entangled.
- The following converse holds: if, for all positive, but not CP maps f, $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geqslant 0$, then ρ_{AB} is separable.
- The transposition map $\Theta(X) = X^t$ is positive, but not CP. Put

$$\mathcal{PPT} = \{\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \,|\, [\mathrm{id}_{d_A} \otimes \Theta_{d_B}](\rho_{AB}) \geqslant 0\}.$$

- The reduction map $R(X) = \mathrm{Tr}(X) \cdot I - X$ is positive, but not CP. Put

$$\mathcal{RED} = \{\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \,|\, [\mathrm{id}_{d_A} \otimes R_{d_B}](\rho_{AB}) \geqslant 0\}.$$

- Both criteria above detect pure entanglement: for $f = \Theta, R$,

$$[\mathrm{id}_{d_A} \otimes f](|\psi\rangle_{AB}\langle\psi|) \geqslant 0 \iff |\psi\rangle \text{ is entangled.}$$

The PPT criterion at work

- Recall the Bell state $\rho_{12} = |\psi\rangle\langle\psi|$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B).$$

- Written as a matrix in $\mathcal{M}^{1,+}_{2\cdot 2}(\mathbb{C})$

$$\rho_{AB} = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) = \frac{1}{2}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

- Partial transposition: transpose each block B_{ij}:

$$[\mathrm{id}_2 \otimes \Theta](\rho_{AB}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- This matrix is no longer positive $\implies$ the state is entangled.

The problem we consider

$$\mathcal{M}^{1,+}(\mathbb{C}^{d_A d_B}) = \{\rho \,|\, \mathrm{Tr}\rho = 1 \text{ and } \rho \geqslant 0\};$$
$$\mathcal{SEP} = \left\{ \sum_i t_i \rho_1(i) \otimes \rho_2(i) \right\};$$
$$\mathcal{PPT} = \{\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \,|\, [\mathrm{id}_{d_A} \otimes \Theta_{d_B}](\rho_{AB}) \geqslant 0\};$$
$$\mathcal{RED} = \{\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \,|\, [\mathrm{id}_{d_A} \otimes R_{d_B}](\rho_{AB}) \geqslant 0\}.$$

Problem

Compare the convex sets

$$\mathcal{SEP} \subset \mathcal{PPT} \subset \mathcal{RED} \subset \mathcal{M}^{1,+}(\mathbb{C}^{d_A d_B}).$$

- For $(d_A, d_B) \in \{(2,2),(2,3),(3,2)\}$ we have $\mathcal{SEP} = \mathcal{PPT}$. In other dimensions, the inclusion $\mathcal{SEP} \subset \mathcal{PPT}$ is strict.
- For $d_B = 2$ we have $\mathcal{PPT} = \mathcal{RED}$. In other dimensions, the inclusion $\mathcal{PPT} \subset \mathcal{RED}$ is strict.

Probability measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- We want to measure volumes of subsets of $\mathcal{M}_d^{1,+}(\mathbb{C})$, with $d = d_A d_B$.
- A first idea would be to use the Lebesgue measure (see $\mathcal{M}_d^{1,+}(\mathbb{C})$ as a compact subset of $\mathcal{M}_d(\mathbb{C})$).
- Another idea: open quantum systems: assume your system Hilbert space $\mathbb{C}^d = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ is coupled to an environment $\mathbb{C}^{d_C}$.
- On the tri-partite system $\mathcal{H}_{ABC} = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_C}$, consider a random pure state $|\psi\rangle_{ABC}$, i.e. a uniform, random point on the unit sphere of the total Hilbert space $\mathcal{H}_{ABC}$.
- Trace out the environment $\mathbb{C}^{d_C}$ to get a *random density matrix*

$$\rho_{AB} = \mathrm{Tr}_C |\psi\rangle\langle\psi|.$$

- These probability measures have been introduced by Zyczkowski and Sommers and they are called the induced measures of parameters $d = d_A d_B$ and $s = d_C$; we denote them by $\mu_{d,s}$.
- Remarkably, the Lebesgue measure is obtained for $d = s$.

Probability measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- Here's an equivalent way of defining the measures $\mu_{d,s}$, in the spirit of Random Matrix Theory.
- Let $X \in \mathcal{M}_{d\times s}(\mathbb{C})$ a rectangular $d \times s$ matrix with i.i.d. complex standard Gaussian entries. Define the random variables
$$W_{d,s} = XX^* \text{ and } \mathcal{M}^{1,+}(\mathbb{C}^d) \ni \rho_{d,s} = \frac{XX^*}{\mathrm{Tr}(XX^*)} = \frac{W_{d,s}}{\mathrm{Tr}\, W_{d,s}}.$$
- The random matrix $W_{d,s}$ is called a Wishart matrix and the distribution of $\rho_{d,s}$ is precisely $\mu_{d,s}$.
- The measure $\mu_{d,s}$ is unitarily invariant: if $\rho \sim \mu_{d,s}$ and U is a random unitary matrix, independent from ρ (e.g. U is constant), then $U\rho U^* \sim \mu_{d,s}$.

Eigenvalues for induced measures

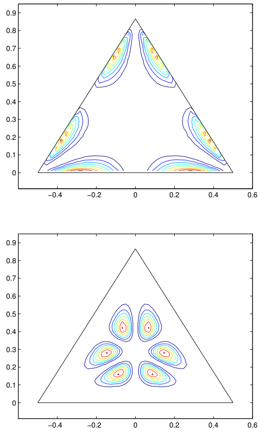


Figure: Induced measure eigenvalue distribution for $(d = 3, s = 3)$, $(d = 3, s = 5)$, $(d = 3, s = 7)$ and $(d = 3, s = 10)$.

Volume of convex sets under the induced measures

- Fix d, and let $C \subset \mathcal{M}^{1,+}(\mathbb{C}^d)$ a convex body, with $\mathrm{I}_d/d \in C^\circ$. Then

$$\lim_{s\to\infty} \mu_{d,s}(C) = 1.$$

In other words, the eigenvalues of a random density matrix $\rho_{AB} \sim \mu_{d,s}$ with d fixed and $s \to \infty$ are close to $1/d$.

Definition

A pair of functions $s_0(d), s_1(d)$ are called a threshold for a family of convex sets $(C_d)_d$ if both conditions below hold

- If $s(d) \lesssim s_0(d)$, then

$$\lim_{d\to\infty} \mu_{d,s(d)}(C_d) = 0;$$

- If $s(d) \gtrsim s_1(d)$, then

$$\lim_{d\to\infty} \mu_{d,s(d)}(C_d) = 1.$$

Thresholds for separability criteria

- In the table below, the threshold functions $s_{0,1}(d)$ are of the form $s_0(d) = s_1(d) = cd$; we put $r = \min(d_A, d_B)$.

Crit. \ Regime	$d_A = d_B \to \infty$	$d_B \to \infty$	$d_A \to \infty$
$\mathcal{SEP}$	∞, $(\sim r \log^q r)$	?	?
$\mathcal{PPT}$	4	$2 + 2\sqrt{1 - \frac{1}{r^2}}$	$2 + 2\sqrt{1 - \frac{1}{r^2}}$
$\mathcal{RED}$	0	0	$\frac{(1+\sqrt{r+1})^2}{r(r-1)}$

- The results in the table above can be interpreted in the following way: for a convex set C having a threshold c, a random density matrix $\rho_{AB} \sim \mu_{d,s}$ will
  - with high probability, belong to C if $s/d > c$
  - with high probability, belong to $\mathcal{M}_d^{1,+}(\mathbb{C}) \setminus C$, if $s/d < c$.
- In other words, the threshold will tell you how large an environment one needs to trace out, in order to obtain random density matrices which are, with high probability, $\mathcal{SEP}$, $\mathcal{PPT}$ or $\mathcal{RED}$.

Proof elements

- The main task is to compute the probability that some random matrices are positive semidefinite or not.
- This is a very difficult computation to perform at fixed Hilbert space dimension; the asymptotic theory is much easier (one or both $d_{A,B} \to \infty$).
- To a selfadjoint matrix $X \in \mathcal{M}_d(\mathbb{C})$, with spectrum $x = (x_1, \ldots, x_d)$, associate its empirical spectral distribution

$$\mu_X = \frac{1}{d} \sum_{i=1}^{d} \delta_{x_i}.$$

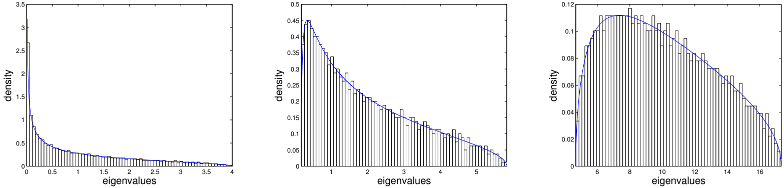
- The probability measure μ_X contains all the information about the spectrum of X.
- A sequence of matrices X_d converges in moments towards a probability measure μ if, for all integer $p \geqslant 1$,

$$\lim_{d\to\infty} \frac{1}{d} \mathrm{Tr}(X_d^p) = \lim_{d\to\infty} \int x^p d\mu_{X_d}(x) = \int x^p d\mu(x).$$

Wishart matrices

Theorem (Marcenko-Pastur)

Let W be a complex Wishart matrix of parameters (d, cd). Then, almost surely with $d \to \infty$, the empirical spectral distribution of $W_{AB}/(cd)$ converges in moments to a free Poisson distribution π_c of parameter c.



(a) (b) (c)

Figure: Eigenvalue distribution for Wishart matrices. In blue, the density of theoretical limiting distribution, π_c. In the three pictures, $d = 1000$, and $c = 1, 2, 10$.

Partial transposition of a Wishart matrix

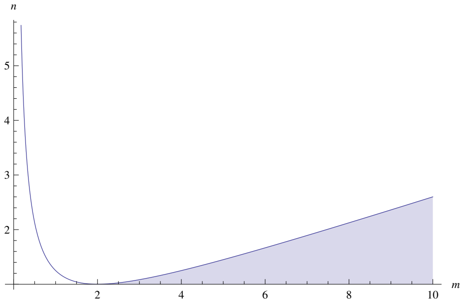
Theorem (Banica, N.)

Let W be a complex Wishart matrix of parameters (dn, dm). Then, almost surely with $d \to \infty$, the empirical spectral distribution of $m[\mathrm{id} \otimes \Theta](W_{AB}/(dm))$ converges in moments to a free difference of free Poisson distributions of respective parameters $m(n \pm 1)/2$.

Corollary

The limiting measure in the previous theorem has positive support iff

$$n \leqslant \frac{m}{4} + \frac{1}{m} \text{ and } m \geqslant 2.$$



Reduction of a Wishart matrix

Theorem (Jivulescu, Lupa, N.)

Let W be a complex Wishart matrix of parameters (dn, cdn). Then, almost surely with $d \to \infty$, the empirical spectral distribution of $[\mathrm{id} \otimes R](W_{AB}/n)$ converges in moments to a compound free Poisson distribution $\pi_{\nu_{n,c}}$ of parameter $\nu_{n,c} = c\delta_{1-n} + c(n^2-1)\delta_1$.

Corollary

The limiting measure in the previous theorem has positive support iff

$$c < \frac{(1+\sqrt{n+1})^2}{n(n-1)}.$$

The free additive convolution of probability measures

- Given two self-adjoint matrices X, Y with spectra x, y, what is the spectrum of $X + Y$?
- In general, a very difficult problem, the answer depends on the relative position of the eigenspaces of X and Y (Horn problem).
- When the size of X, Y is large, and the eigenvectors are in general position, free probability theory [Voiculescu, '80s] gives the answer.
- Free additive convolution (or free sum) of two compactly supported probability distributions μ, ν: sample $x, y \in \mathbb{R}^n$ from μ, ν and consider

$$Z = \mathrm{diag}(x) + U\mathrm{diag}(y)U^*,$$

  where U is a $d \times d$ Haar unitary random matrix. Then, as $d \to \infty$, the empirical eigenvalue distribution of Z converges to a probability measure denoted by $\mu \boxplus \nu$.
- The operation $\boxplus$ is called free additive convolution, and it can be computed via the so-called $\mathcal{R}$-transform (a kind of Fourier transform in the free world)

The free Poisson distribution

- The free Poisson distribution of parameter $c > 0$:

$$\pi_c = \max(1-c,0)\delta_0 + \frac{\sqrt{4c-(x-1-c)^2}}{2\pi x}\mathbf{1}_{[(1-\sqrt{c})^2,(1+\sqrt{c})^2]}(x)\,dx.$$

- The measure π_c is the limit eigenvalue distribution of a rescaled density matrix from the induced ensemble $\rho_{d,cd}$ (d large).
- One can show a free Poisson Central Limit Theorem:

$$\lim_{n\to\infty}\left[\left(1-\frac{c}{n}\right)\delta_0 + \frac{c}{n}\delta_1\right]^{\boxplus n} = \pi_c.$$

- The free compound Poisson measure of parameter ν is defined via a generalized free Poisson central limit theorem

$$\lim_{n\to\infty}\left[\left(1-\frac{\nu(\mathbb{R})}{n}\right)\delta_0 + \frac{1}{n}\nu\right]^{\boxplus n} =: \pi_\nu.$$

- Its support and probability density are much harder to compute.

Thank you !

1. Banica, N. - *Asymptotic eigenvalue distributions of block-transposed Wishart matrices* - J. Theoret. Probab. 26 (2013), 855-869
2. Jivulescu, Lupa, N. - *On the reduction criterion for random quantum states* - `arXiv:1402.4292`