Positive and completely positive maps via free additive powers of probability measures

Ion Nechita

TU München and CNRS

joint work in progress with Benoit Collins (Kyoto) and Patrick Hayden (Stanford)

Bedlewo, July 9th 2014

Entanglement in Quantum Information Theory

- Quantum states with n degrees of freedom are described by density matrices

$$\rho \in \mathbb{M}_n^{1,+} = \mathrm{End}^{1,+}(\mathbb{C}^n); \qquad \mathrm{Tr}\rho = 1 \text{ and } \rho \geq 0$$

- Two quantum systems: $\rho_{12} \in \mathrm{End}^{1,+}(\mathbb{C}^m \otimes \mathbb{C}^n) = \mathbb{M}_{mn}^{1,+}$
- A state ρ_{12} is called separable if it can be written as a convex combination of product states

$$\rho_{12} \in \mathcal{SEP} \iff \rho_{12} = \sum_i t_i \rho_1(i) \otimes \rho_2(i),$$

where $t_i \geq 0$, $\sum_i t_i = 1$, $\rho_1(i) \in \mathbb{M}_m^{1,+}$, $\rho_2(i) \in \mathbb{M}_n^{1,+}$

- Equivalently, $\mathcal{SEP} = \mathrm{conv}\left[\mathbb{M}_m^{1,+} \otimes \mathbb{M}_n^{1,+}\right]$
- Non-separable states are called entangled

More on entanglement - pure states

- Separable rank one (pure) states $\rho_{12} = P_{e \otimes f} = P_e \otimes P_f$.
- Bell state or maximally entangled state $\rho_{12} = P_{\mathrm{Bell}}$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni \mathrm{Bell} = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2) \neq x \otimes y.$$

- For rank one quantum states, entanglement can be detected and quantified by the entropy of entanglement

$$E_{\mathrm{ent}}(P_x) = H(s(x)) = -\sum_{i=1}^{\min(m,n)} s_i(x) \log s_i(x),$$

where $x \in \mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{M}_{m \times n}(\mathbb{C})$ is seen as a $m \times n$ matrix and $s_i(x)$ are its singular values.
- A pure state $x \in \mathbb{C}^m \otimes \mathbb{C}^n$ is separable $\iff$ $E_{\mathrm{ent}}(P_x) = 0$.

Separability criteria

- Let $\mathcal{A}$ be a C^* algebra. A map $f : \mathbb{M}_n \to \mathcal{A}$ is called
  - positive if $A \geq 0 \implies f(A) \geq 0$;
  - completely positive (CP) if $\mathrm{id}_k \otimes f$ is positive for all $k \geq 1$ ($k = n$ is enough).
- Let $f : \mathbb{M}_n \to \mathcal{A}$ be a completely positive map. Then, for every state $\rho_{12} \in \mathbb{M}_{mn}^{1,+}$, one has $[\mathrm{id}_m \otimes f](\rho_{12}) \geq 0$.
- Let $f : \mathbb{M}_n \to \mathcal{A}$ be a positive map. Then, for every separable state $\rho_{12} \in \mathbb{M}_{mn}^{1,+}$, one has $[\mathrm{id}_m \otimes f](\rho_{12}) \geq 0$.
  - ρ_{12} separable $\implies \rho_{12} = \sum_i t_i \rho_1(i) \otimes \rho_2(i)$.
  - $[\mathrm{id}_m \otimes f](\rho_{12}) = \sum_i t_i \rho_1(i) \otimes f(\rho_2(i))$.
  - For all i, $([\rho_2(i)) \geq 0$, so $[\mathrm{id}_m \otimes f](\rho_{12}) \geq 0$.
- Hence, positive, but not CP maps f provide sufficient entanglement criteria: if $[\mathrm{id}_m \otimes f](\rho_{12}) \not\geq 0$, then ρ_{12} is entangled.
- Moreover, if $[\mathrm{id}_m \otimes f](\rho_{12}) \geq 0$ for all positive, but not CP maps $f : \mathbb{M}_n \to \mathbb{M}_m$, then ρ_{12} is separable.

Positive Partial Transpose matrices

- The transposition map $\mathrm{t} : A \mapsto A^t$ is positive, but not CP. Define the convex set

$$\mathcal{PPT} = \{\rho_{12} \in \mathbb{M}_{mn}^{1,+} \mid [\mathrm{id}_m \otimes \mathrm{t}_n](\rho_{12}) \geq 0\}.$$

- For $(m, n) \in \{(2,2), (2,3)\}$ we have $\mathcal{SEP} = \mathcal{PPT}$. In other dimensions, the inclusion $\mathcal{SEP} \subset \mathcal{PPT}$ is strict.
- Low dimensions are special because every positive map $f : \mathbb{M}_2 \to \mathbb{M}_{2/3}$ is decomposable:

$$f = g_1 + g_2 \circ \mathrm{t},$$

with $g_{1,2}$ completely positive. Among all decomposable maps, the transposition criterion is the strongest.

The Choi matrix of a map

- For any n, recall that the maximally entangled state is the orthogonal projection onto

$$\mathbb{C}^n \otimes \mathbb{C}^n \ni \mathrm{Bell} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i \otimes e_i.$$

- To any map $f : \mathbb{M}_n \to \mathcal{A}$, associate its Choi matrix

$$C_f = [\mathrm{id}_n \otimes f](P_{\mathrm{Bell}}) \in \mathbb{M}_n \otimes \mathcal{A}.$$

- Equivalently, if E_{ij} are the matrix units in $\mathbb{M}_n$, then

$$C_f = \sum_{i,j=1}^{n} E_{ij} \otimes f(E_{ij}).$$

Theorem (Choi '72)

A map $f : \mathbb{M}_n \to \mathcal{A}$ is CP iff its Choi matrix C_f is positive.

The Choi-Jamiołkowski isomorphism

- Recall (from now on $\mathcal{A} = \mathbb{M}_d$)

$$C_f = [\mathrm{id}_n \otimes f](P_{\mathrm{Bell}}) = \sum_{i,j=1}^{n} E_{ij} \otimes f(E_{ij}) \in \mathbb{M}_n \otimes \mathbb{M}_d.$$

- The map $f \mapsto C_f$ is called the Choi-Jamiołkowski isomorphism.
- It sends:
  1. All linear maps to all operators;
  2. Hermicity preserving maps to hermitian operators;
  3. Entanglement breaking maps to separable quantum states;
  4. Unital maps to operators with unit left partial trace ($[\mathrm{Tr} \otimes \mathrm{id}] C_f = \mathrm{I}_d$);
  5. Trace preserving maps to operators with unit left partial trace ($[\mathrm{id} \otimes \mathrm{Tr}] C_f = \mathrm{I}_n$).

Intermediate positivity notions

- A map $f : \mathbb{M}_n \to \mathcal{A}$ is called k-positive if $\mathrm{id}_k \otimes f$ is positive.
- A matrix $C \in \mathbb{M}_{nd}$ is called k-positive if $\langle x, Cx \rangle \geq 0$ for all vectors $x \in \mathbb{C}^n \otimes \mathbb{C}^d$ of rank at most k.
- In particular, C is 1-positive (or block-positive) if

$$\forall x \in \mathbb{C}^n, \forall y \in \mathbb{C}^d \qquad \langle x \otimes y, C \cdot x \otimes y \rangle \geq 0.$$

Theorem

A map $f : \mathbb{M}_n \to \mathcal{A}$ is k-positive iff its Choi matrix C_f is k-positive. In particular, f is positive iff C_f is block-positive.

Random Choi matrices

- Let μ be a compactly supported probability measure on $\mathbb{R}$. For each d we introduce a real valued diagonal matrix X_d of $\mathbb{M}_n \otimes \mathbb{M}_d$ whose eigenvalue counting distribution converges to μ and whose extremal eigenvalues converge to the respective extrema of the support of μ.
- Let U_d be a random Haar unitary matrix in the unitary group $\mathcal{U}_{nd}$, and $f_\mu^{(d)} : \mathbb{M}_n \to \mathbb{M}_d$ be the map whose Choi matrix is $U_d X_d U_d^*$.

Theorem

Under the above assumptions, if $\mathrm{supp}(\mu^{\boxplus n/k}) \subset (0, \infty)$ *then, almost surely as* $d \to \infty$*, the map* $f_\mu^{(d)}$ *is* k*-positive. On the other hand, if* $\mathrm{supp}(\mu^{\boxplus n/k}) \cap (-\infty, 0) \neq \emptyset$ *then, almost surely as* $d \to \infty$*,* $f_\mu^{(d)}$ *is not* k*-positive.*

Proof ingredients

Let $f_\mu^{(d)} : \mathbb{M}_n \to \mathbb{M}_d$ be the map whose Choi matrix is $U_d X_d U_d^*$.

Theorem

If $\operatorname{supp}(\mu^{\boxplus n/k}) \subset (0, \infty)$ then, almost surely as $d \to \infty$, the map $f_\mu^{(d)}$ is k-positive. If $\operatorname{supp}(\mu^{\boxplus n/k}) \cap (-\infty, 0) \neq \emptyset$ then, almost surely as $d \to \infty$, $f_\mu^{(d)}$ is not k-positive.

Proposition

A map f is k-positive iff for any self-adjoint projection $P \in \mathbb{M}_n$ of rank k, the operator $(P \otimes I_d) C_f (P \otimes I_d)$ is positive.

Proposition (Nica and Speicher)

Let x, p be free elements in a ncps $(\mathcal{M}, \tau)$ and assume that p is a selfadjoint projection of rank $\tau(p) = 1/t$ $(t \geq 1)$ and that x has distribution μ. Then, the distribution of $t^{-1} pxp$ inside the contracted ncps $(p\mathcal{M}p, \tau(p \cdot p))$ is $\mu^{\boxplus t}$

Maps associated to probability measures

- Let μ be a compactly supported probability measure on $\mathbb{R}$.
- The vN algebra $L^\infty(\mathbb{R}, \mu)$, endowed with the expectation trace $\mathbb{E}$ is a non-commutative probability space. Let $X \in L^\infty(\mathbb{R}, \mu)$ be the identity map $x \mapsto x$.
- Consider the vN ncps free product $(\tilde{\mathcal{M}}, \mathrm{tr} * \mathbb{E}) = (\mathbb{M}_n, \mathrm{tr}) * (L^\infty(\mathbb{R}, \mu), \mathbb{E})$.
- Finally, let $(\mathcal{M}, \tau)$ be the contracted vN ncps $\mathcal{M} = E_{11}\tilde{\mathcal{M}}E_{11}$.
- Define

$$\begin{aligned} f_\mu : \mathbb{M}_n &\to \mathcal{M} \\ E_{ij} &\mapsto E_{1i} X E_{j1} \end{aligned}$$

Theorem

The map f_μ is k-positive iff $\mathrm{supp}(\mu^{\boxplus n/k}) \subseteq [0, \infty)$.

Example: semicircular measures

- Let $s_{a,\sigma}$ be the semi-circle distribution of mean a and variance σ^2, having support $[a - 2\sigma, a + 2\sigma]$.
- We have $s_{a,\sigma}^{\boxplus n/k} = s_{an/k, \sigma\sqrt{n/k}}$, with support $\mathrm{supp}(s_{a,\sigma}^{\boxplus n/k}) = [an/k - 2\sigma\sqrt{n/k}, an/k + 2\sigma\sqrt{n/k}]$.

Theorem

Let n be an integer and a, σ positive parameters. The map $f_{a,\sigma} : \mathbb{M}_n \to \mathcal{M}$ associated to a semi-circular distribution $s_{a,\sigma}$ is k-positive iff $k \leq 4n\sigma^2/a^2$. In particular, for any n and any $k < n$, there exist parameters $a, \sigma > 0$ such that the above map is k-positive but not $(k+1)$-positive.

Example: semicircular measures, $t = n/k$

Semicircular measures vs. PPT

- We show next that the maps $f_{a,\sigma}$ detect some PPT and entangled states. Importantly, the states detected are correlated to the Choi matrix defining $f_{a,\sigma}$
- Consider (normalized) i.i.d. GUE matrices $S_{ij}, S'_{ij} \in \mathbb{M}_d^{sa}$ and define, for $\alpha \in (-1,1)$, the selfadjoint test matrix $X_d \in \mathbb{M}_n \otimes \mathbb{M}_d$, with the following blocks:
  - diagonal blocks $X_d(i,i) = 2\sqrt{2}\sqrt{n}\mathrm{I}_d - \alpha\sqrt{2}S_{ii}$ for $1 \leq i \leq n$;
  - off-diagonal blocks $X_d(i,j) = \alpha(-\bar{S}_{ij} + \sqrt{-1}\bar{S}'_{ij})$, for $1 \leq i < j \leq n$.
- We have $X_d = \sqrt{2n}(2\mathrm{I}_{dn} + \alpha Y_d)$, where $Y_d \in \mathbb{M}_{dn}^{sa}$ is a GUE.
- Almost surely, as $d \to \infty$, X_d is positive semidefinite and PPT (since the GUE ensemble is invariant under partial transposition)
- Fix a small $\varepsilon > 0$ and let $f_d : \mathbb{M}_n \to \mathbb{M}_d$ be the linear map whose Choi matrix C_d has blocks
  - diagonal blocks $C_d(i,i) = (2\sqrt{2} + \varepsilon)\mathrm{I}_d + \sqrt{2}S_{ii}$ for $1 \leq i \leq n$;
  - off-diagonal blocks $C_d(i,j) = S_{ij} + \sqrt{-1}S'_{ij}$, for $1 \leq i < j \leq n$.
- Using our main result, we check easily that, almost surely as $d \to \infty$, the maps f_d are positive (1-positive).

Semicircular measures vs. PPT

- Recall the maximally entangled state $\mathrm{Bell}_d \in \mathbb{C}^d \otimes \mathbb{C}^d$

$$\mathrm{Bell}_d = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} e_i \otimes e_i,$$

  for some orthonormal basis $\{e_i\}$ of $\mathbb{C}^d$.

- A direct computation shows that, almost surely as $d \to \infty$,

$$\langle \mathrm{Bell}_d, [f_d \otimes \mathrm{id}_d](X_d) \cdot \mathrm{Bell}_d \rangle \sim n(2\sqrt{2n}(2\sqrt{2}+\varepsilon)-2\alpha)-2n(n-1)\alpha.$$

Theorem

Given $\alpha \in (-1,1)$, there exists $\varepsilon > 0$ small enough, such that, as soon as $n\alpha^2 > 16$, the matrix $[f_d \otimes \mathrm{id}_d](X_d)$ is almost surely not positive semidefinite, as $d \to \infty$, and thus X_d is entangled and PPT. In particular, the maps f_d are indecomposable.