Positive and completely positive maps via free additive powers of probability measures

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Quantum Information Theory, Quantum Computing

- New branches of $\{\text{Physics}, \text{Computer Science}, \text{Mathematics}\}$ dealing with quantum information
- Quantum information $=$ information held in a quantum physical system
- Basic idea: replace $\{0, 1\}$ with $\mathbb{C}^2 = \mathrm{span}\{|0\rangle, |1\rangle\}$, the state space of a two-level quantum system (qubit)
- Two qubits: $\mathbb{C}^2 \otimes \mathbb{C}^2 \rightsquigarrow$ entanglement
- Shows great promise:
  1. Secure transmission of data, protocol security guaranteed by the laws of nature
  2. Fast integer factorization $\rightsquigarrow$ current algorithms (RSA, etc) obsolete
  3. Fast database search
  4. Fast simulation of quantum systems

Entanglement in Quantum Information Theory

- Quantum states with n degrees of freedom are described by density matrices

$$\rho \in \mathbb{M}_n^{1,+} = \mathrm{End}^{1,+}(\mathbb{C}^n); \qquad \mathrm{Tr}\rho = 1 \text{ and } \rho \geq 0$$

- Two quantum systems: $\rho_{12} \in \mathrm{End}^{1,+}(\mathbb{C}^m \otimes \mathbb{C}^n) = \mathbb{M}_{mn}^{1,+}$
- A state ρ_{12} is called separable if it can be written as a convex combination of product states

$$\rho_{12} \in \mathcal{SEP} \iff \rho_{12} = \sum_i t_i \rho_1(i) \otimes \rho_2(i),$$

  where $t_i \geq 0$, $\sum_i t_i = 1$, $\rho_1(i) \in \mathbb{M}_m^{1,+}$, $\rho_2(i) \in \mathbb{M}_n^{1,+}$
- Equivalently, $\mathcal{SEP} = \mathrm{conv}\left[\mathbb{M}_m^{1,+} \otimes \mathbb{M}_n^{1,+}\right]$
- Non-separable states are called entangled

More on entanglement - pure states

- Separable rank one (pure) states $\rho_{12} = P_{e \otimes f} = P_e \otimes P_f$.
- Bell state or maximally entangled state $\rho_{12} = P_{\mathrm{Bell}}$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni \mathrm{Bell} = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2) \neq x \otimes y.$$

- For rank one quantum states, entanglement can be detected and quantified by the entropy of entanglement

$$E_{\mathrm{ent}}(P_x) = H(s(x)) = -\sum_{i=1}^{\min(m,n)} s_i(x) \log s_i(x),$$

  where $x \in \mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{M}_{m \times n}(\mathbb{C})$ is seen as a $m \times n$ matrix and $s_i(x)$ are its singular values.
- A pure state $x \in \mathbb{C}^m \otimes \mathbb{C}^n$ is separable $\iff$ $E_{\mathrm{ent}}(P_x) = 0$.

Separability criteria

- Let $\mathcal{A}$ be a C^* algebra. A map $f : \mathbb{M}_n \to \mathcal{A}$ is called
  - positive if $A \geq 0 \implies f(A) \geq 0$;
  - completely positive (CP) if $\mathrm{id}_k \otimes f$ is positive for all $k \geq 1$ ($k = n$ is enough).
- Let $f : \mathbb{M}_n \to \mathcal{A}$ be a completely positive map. Then, for every state $\rho_{12} \in \mathbb{M}_{mn}^{1,+}$, one has $[\mathrm{id}_m \otimes f](\rho_{12}) \geq 0$.
- Let $f : \mathbb{M}_n \to \mathcal{A}$ be a positive map. Then, for every separable state $\rho_{12} \in \mathbb{M}_{mn}^{1,+}$, one has $[\mathrm{id}_m \otimes f](\rho_{12}) \geq 0$.
  - ρ_{12} separable $\implies \rho_{12} = \sum_i t_i \rho_1(i) \otimes \rho_2(i)$.
  - $[\mathrm{id}_m \otimes f](\rho_{12}) = \sum_i t_i \rho_1(i) \otimes f(\rho_2(i))$.
  - For all i, $([\rho_2(i)) \geq 0$, so $[\mathrm{id}_m \otimes f](\rho_{12}) \geq 0$.
- Hence, positive, but not CP maps f provide sufficient entanglement criteria: if $[\mathrm{id}_m \otimes f](\rho_{12}) \not\geq 0$, then ρ_{12} is entangled.
- Moreover, if $[\mathrm{id}_m \otimes f](\rho_{12}) \geq 0$ for all positive, but not CP maps $f : \mathbb{M}_n \to \mathbb{M}_m$, then ρ_{12} is separable.

Positive Partial Transpose matrices

- The transposition map $\mathrm{t} : A \mapsto A^t$ is positive, but not CP. Define the convex set

$$\mathcal{PPT} = \{\rho_{12} \in \mathbb{M}_{mn}^{1,+} \mid [\mathrm{id}_m \otimes \mathrm{t}_n](\rho_{12}) \geq 0\}.$$

- For $(m, n) \in \{(2,2),(2,3)\}$ we have $\mathcal{SEP} = \mathcal{PPT}$. In other dimensions, the inclusion $\mathcal{SEP} \subset \mathcal{PPT}$ is strict.
- Low dimensions are special because every positive map $f : \mathbb{M}_2 \to \mathbb{M}_{2/3}$ is decomposable:

$$f = g_1 + g_2 \circ \mathrm{t},$$

with $g_{1,2}$ completely positive. Among all decomposable maps, the transposition criterion is the strongest.

The PPT criterion at work

- Recall the Bell state $\rho_{12} = P_{\mathrm{Bell}}$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni \mathrm{Bell} = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2).$$

- Written as a matrix in $\mathbb{M}_{2\cdot 2}^{1,+}$

$$\rho_{12} = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) = \frac{1}{2}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

- Partial transposition: transpose each block B_{ij}:

$$\rho_{12}^{\Gamma} = [\mathrm{id}_2 \otimes \mathrm{t}_2](\rho_{12}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- This matrix is no longer positive $\Longrightarrow$ the state is entangled.

The Choi matrix of a map

- For any n, recall that the maximally entangled state is the orthogonal projection onto

$$\mathbb{C}^n \otimes \mathbb{C}^n \ni \mathrm{Bell} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i \otimes e_i.$$

- To any map $f : \mathbb{M}_n \to \mathcal{A}$, associate its Choi matrix

$$C_f = [\mathrm{id}_n \otimes f](P_{\mathrm{Bell}}) \in \mathbb{M}_n \otimes \mathcal{A}.$$

- Equivalently, if E_{ij} are the matrix units in $\mathbb{M}_n$, then

$$C_f = \sum_{i,j=1}^{n} E_{ij} \otimes f(E_{ij}).$$

Theorem (Choi '72)

A map $f : \mathbb{M}_n \to \mathcal{A}$ is CP iff its Choi matrix C_f is positive.

The Choi-Jamiołkowski isomorphism

- Recall (from now on $\mathcal{A} = \mathbb{M}_d$)

$$C_f = [\mathrm{id}_n \otimes f](P_{\mathrm{Bell}}) = \sum_{i,j=1}^{n} E_{ij} \otimes f(E_{ij}) \in \mathbb{M}_n \otimes \mathbb{M}_d.$$

- The map $f \mapsto C_f$ is called the Choi-Jamiołkowski isomorphism.
- It sends:
  1. All linear maps to all operators;
  2. Hermicity preserving maps to hermitian operators;
  3. Entanglement breaking maps to separable quantum states;
  4. Unital maps to operators with unit left partial trace ($[\mathrm{Tr} \otimes \mathrm{id}] C_f = \mathrm{I}_d$);
  5. Trace preserving maps to operators with unit left partial trace ($[\mathrm{id} \otimes \mathrm{Tr}] C_f = \mathrm{I}_n$).

Intermediate positivity notions

- A map $f : \mathbb{M}_n \to \mathcal{A}$ is called k-positive if $\mathrm{id}_k \otimes f$ is positive.
- A matrix $C \in \mathbb{M}_{nd}$ is called k-positive if $\langle x, Cx \rangle \geq 0$ for all vectors $x \in \mathbb{C}^n \otimes \mathbb{C}^d$ of rank at most k.
- In particular, C is 1-positive (or block-positive) if

$$\forall x \in \mathbb{C}^n, \forall y \in \mathbb{C}^d \qquad \langle x \otimes y, C \cdot x \otimes y \rangle \geq 0.$$

Theorem

A map $f : \mathbb{M}_n \to \mathcal{A}$ is k-positive iff its Choi matrix C_f is k-positive. In particular, f is positive iff C_f is block-positive.

Random Choi matrices

- Let μ be a compactly supported probability measure on $\mathbb{R}$. For each d we introduce a real valued diagonal matrix X_d of $\mathbb{M}_n \otimes \mathbb{M}_d$ whose eigenvalue counting distribution converges to μ and whose extremal eigenvalues converge to the respective extrema of the support of μ.
- Let U_d be a random Haar unitary matrix in the unitary group $\mathcal{U}_{nd}$, and $f_\mu^{(d)} : \mathbb{M}_n \to \mathbb{M}_d$ be the map whose Choi matrix is $U_d X_d U_d^*$.

Theorem

Under the above assumptions, if $\mathrm{supp}(\mu^{\boxplus n/k}) \subset (0, \infty)$ *then, almost surely as* $d \to \infty$, *the map* $f_\mu^{(d)}$ *is* k*-positive. On the other hand, if* $\mathrm{supp}(\mu^{\boxplus n/k}) \cap (-\infty, 0) \neq \emptyset$ *then, almost surely as* $d \to \infty$, $f_\mu^{(d)}$ *is not* k*-positive.*

Proof ingredients

Let $f_\mu^{(d)} : \mathbb{M}_n \to \mathbb{M}_d$ be the map whose Choi matrix is $U_d X_d U_d^*$.

Theorem

If $\mathrm{supp}(\mu^{\boxplus n/k}) \subset (0, \infty)$ then, almost surely as $d \to \infty$, the map $f_\mu^{(d)}$ is k-positive. If $\mathrm{supp}(\mu^{\boxplus n/k}) \cap (-\infty, 0) \neq \emptyset$ then, almost surely as $d \to \infty$, $f_\mu^{(d)}$ is not k-positive.

Proposition

A map f is k-positive iff for any self-adjoint projection $P \in \mathbb{M}_n$ of rank k, the operator $(P \otimes I_d)C_f(P \otimes I_d)$ is positive.

Proposition (Nica and Speicher)

Let x, p be free elements in a ncps $(\mathcal{M}, \tau)$ and assume that p is a selfadjoint projection of rank $\tau(p) = 1/t$ $(t \geq 1)$ and that x has distribution μ. Then, the distribution of $t^{-1}pxp$ inside the contracted ncps $(p\mathcal{M}p, \tau(p \cdot p))$ is $\mu^{\boxplus t}$

Maps associated to probability measures

- Let μ be a compactly supported probability measure on $\mathbb{R}$.
- The vN algebra $L^\infty(\mathbb{R}, \mu)$, endowed with the expectation trace $\mathbb{E}$ is a non-commutative probability space. Let $X \in L^\infty(\mathbb{R}, \mu)$ be the identity map $x \mapsto x$.
- Consider the vN ncps free product $(\tilde{\mathcal{M}}, \mathrm{tr} * \mathbb{E}) = (\mathbb{M}_n, \mathrm{tr}) * (L^\infty(\mathbb{R}, \mu), \mathbb{E})$.
- Finally, let $(\mathcal{M}, \tau)$ be the contracted vN ncps $\mathcal{M} = E_{11}\tilde{\mathcal{M}}E_{11}$.
- Define

$$\begin{aligned} f_\mu : \mathbb{M}_n &\to \mathcal{M} \\ E_{ij} &\mapsto E_{1i} X E_{j1} \end{aligned}$$

Theorem

The map f_μ is k-positive iff $\mathrm{supp}(\mu^{\boxplus n/k}) \subseteq [0, \infty)$.

Example: semicircular measures

- Let $s_{a,\sigma}$ be the semi-circle distribution of mean a and variance σ^2, having support $[a - 2\sigma, a + 2\sigma]$.
- We have $s_{a,\sigma}^{\boxplus n/k} = s_{an/k, \sigma\sqrt{n/k}}$, with support $\mathrm{supp}(s_{a,\sigma}^{\boxplus n/k}) = [an/k - 2\sigma\sqrt{n/k}, an/k + 2\sigma\sqrt{n/k}]$.

Theorem

Let n be an integer and a, σ positive parameters. The map $f_{a,\sigma} : \mathbb{M}_n \to \mathcal{M}$ associated to a semi-circular distribution $s_{a,\sigma}$ is k-positive iff $k \leq 4n\sigma^2/a^2$. In particular, for any n and any $k < n$, there exist parameters $a, \sigma > 0$ such that the above map is k-positive but not $(k+1)$-positive.

Semicircular measures vs. PPT

- We show next that the maps $f_{a,\sigma}$ detect some PPT and entangled states. Importantly, the states detected are correlated to the Choi matrix defining $f_{a,\sigma}$
- Consider (normalized) i.i.d. GUE matrices $S_{ij}, S'_{ij} \in \mathbb{M}_d^{sa}$ and define, for $\alpha \in (-1,1)$, the selfadjoint test matrix $X_d \in \mathbb{M}_n \otimes \mathbb{M}_d$, with the following blocks:
  - diagonal blocks $X_d(i,i) = 2\sqrt{2}\sqrt{n}\mathrm{I}_d - \alpha\sqrt{2}S_{ii}$ for $1 \leq i \leq n$;
  - off-diagonal blocks $X_d(i,j) = \alpha(-\bar{S}_{ij} + \sqrt{-1}\bar{S}'_{ij})$, for $1 \leq i < j \leq n$.
- We have $X_d = \sqrt{2n}(2\mathrm{I}_{dn} + \alpha Y_d)$, where $Y_d \in \mathbb{M}_{dn}^{sa}$ is a GUE.
- Almost surely, as $d \to \infty$, X_d is positive semidefinite and PPT (since the GUE ensemble is invariant under partial transposition)
- Fix a small $\varepsilon > 0$ and let $f_d : \mathbb{M}_n \to \mathbb{M}_d$ be the linear map whose Choi matrix C_d has blocks
  - diagonal blocks $C_d(i,i) = (2\sqrt{2} + \varepsilon)\mathrm{I}_d + \sqrt{2}S_{ii}$ for $1 \leq i \leq n$;
  - off-diagonal blocks $C_d(i,j) = S_{ij} + \sqrt{-1}S'_{ij}$, for $1 \leq i < j \leq n$.
- Using our main result, we check easily that, almost surely as $d \to \infty$, the maps f_d are positive (1-positive).

Semicircular measures vs. PPT

- Recall the maximally entangled state $\mathrm{Bell}_d \in \mathbb{C}^d \otimes \mathbb{C}^d$

$$\mathrm{Bell}_d = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} e_i \otimes e_i,$$

for some orthonormal basis $\{e_i\}$ of $\mathbb{C}^d$.

- A direct computation shows that, almost surely as $d \to \infty$,

$$\langle \mathrm{Bell}_d, [f_d \otimes \mathrm{id}_d](X_d) \cdot \mathrm{Bell}_d \rangle \sim n(2\sqrt{2n}(2\sqrt{2}+\varepsilon)-2\alpha)-2n(n-1)\alpha.$$

Theorem

Given $\alpha \in (-1,1)$, there exists $\varepsilon > 0$ small enough, such that, as soon as $n\alpha^2 > 16$, the matrix $[f_d \otimes \mathrm{id}_d](X_d)$ is almost surely not positive semidefinite, as $d \to \infty$, and thus X_d is entangled and PPT. In particular, the maps f_d are indecomposable.