Quantum channels with polytopic images

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Outline

1. Characterization of quantum channels having polytopic image
2. Characterization of quantum channels satisfying a stronger form of additivity

Introduction

A convex set $C \subseteq \mathbb{R}^d$ is said to be a polytope if it is the convex hull of a finite number of points $C = \mathrm{conv}(x_1, \ldots, x_k)$. Equivalently, a polytope is the bounded intersection of a finite number of half-spaces.

A quantum channel is a linear map $T : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ that is completely positive and trace preserving. In particular, T sends quantum states to quantum states.

The image of a quantum channel $\mathrm{Im}(T)$ is a convex, compact subset of $\mathcal{S}_n := \{\rho \in \mathcal{M}_n \, : \, \rho \geq 0 \text{ and } \mathrm{Tr}\rho = 1\}$.

Problem

Characterize channels T for which $\mathrm{Im}(T)$ is a polytope.

Examples

- Classical-classical channels: $T : \mathcal{M}_d \to \mathcal{M}_d$,

$$T(\rho) = \sum_{i=1}^{d} \langle i|\rho|i\rangle \cdot |i\rangle\langle i|.$$

  Here, $\mathrm{Im}(T) = \mathrm{diag}(\mathcal{M}_d) = \mathrm{conv}(|i\rangle\langle i|)$.

- Classical-quantum channels: $T : \mathcal{M}_d \to \mathcal{M}_n$,

$$T(\rho) = \sum_{i=1}^{d} \langle i|\rho|i\rangle \cdot \sigma_i,$$

  for some $\sigma_i \in \mathcal{S}_n$. Here, $\mathrm{Im}(T) = \mathrm{conv}(\sigma_1, \ldots, \sigma_d)$.

- Essentially classical-quantum channels: $T : \mathcal{M}_d \to \mathcal{M}_n$,

$$T(\rho) = \sum_{i=1}^{d} \mathrm{Tr}(M_i \rho) \cdot \sigma_i,$$

  for some $\sigma_i \in \mathcal{S}_n$ and a POVM $\{M_i\}$, with the property that $\|M_i\| = 1$, for all i. As before, $\mathrm{Im}(T) = \mathrm{conv}(\sigma_1, \ldots, \sigma_d)$.

A non-example

Note that all examples above were entanglement breaking: they can be written as

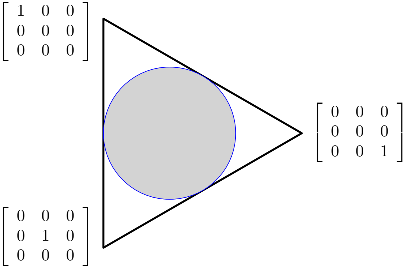
$$T(\rho) = \sum_{i=1}^{d} \mathrm{Tr}(M_i \rho) \cdot \sigma_i,$$

for arbitrary output states σ_i and a general POVM $\{M_i\}$.

But not all entanglement breaking channels have polytopic images. Consider $T : \mathcal{M}_2 \to \mathcal{M}_3$,

$$T(\rho) = \sum_{i=1}^{3} \mathrm{Tr}(M_i \rho) |i\rangle\langle i|,$$

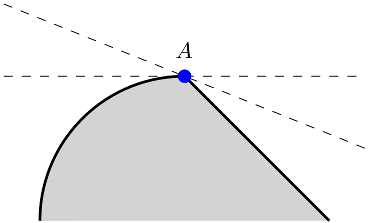
where the POVM operators $M_i \in \mathcal{M}_2$ are defined by $M_i = 2/3 P_{\omega^i}$, where P_{ω^i} is the orthogonal projection on the complex number $\omega^i \in \mathbb{C}$ (seen as a vector in $\mathbb{R}^2 \subset \mathbb{C}^2$) and ω is a third root of unity.



Vertices of convex sets

Definition

A point x on the boundary of a convex set C is called a vertex if the intersection of all supporting hyperplanes of C at x is the set $\{x\}$. In particular, x is then an extreme point of C.



For convex polytopes, the sets of vertices and extreme points coincide. On the other hand, the set of all quantum states $\mathcal{S}_n$ has no vertices.

Main result (quantum channels with polytopic images)

Theorem

Let $T : \mathcal{M}_d \to \mathcal{M}_n$ be a quantum channel whose image is a convex polytope with k vertices $\{\sigma_i\}_{i=1}^k$. Then, there exists a subspace $V \subseteq \mathbb{C}^d$ such that

$$T(\rho) = T_1(\rho_V) + T_2(\rho_{V^\perp})$$

where T_1 is an essentially classical-quantum channel ($\|M_i\| = 1$)

$$T_1(\rho) = \sum_{i=1}^{k} \mathrm{Tr}(M_i \rho)\sigma_i$$

and T_2 is another quantum channel whose image is hidden behind the image of T_1: $\mathrm{Im}(T_2) \subseteq \mathrm{Im}(T_1)$ and, for all i, $\sigma_i \notin \mathrm{Im}(T_2)$.

Conversely, every map of this form has polytopic image with vertices $\{\sigma_i\}$.

Proof idea: qubit restrictions

The image of a qubit channel $S : \mathcal{M}_2 \to \mathcal{M}_2$ is an (eventually degenerate) ellipsoid. The image set $\mathrm{Im}(S)$ has vertices if and only if it is a segment $[\sigma_1, \sigma_2]$ with

$$S(\rho) = \langle x|\rho|x\rangle \sigma_1 + \langle x^\perp|\rho|x^\perp\rangle \sigma_2$$

or it is a single point σ.

Lemma

Consider a quantum channel $T : \mathcal{M}_d \to \mathcal{M}_n$ and two different pure states $x, y \in \mathbb{C}^d$ such that $T(|x\rangle\langle x|)$ is a vertex of $\mathrm{Im}(T)$. If H is the two-dimensional subspace of $\mathbb{C}^d$ spanned by x and y, then the restriction T_H of T to $\mathrm{End}(H)$ is of the form

$$T_H(\rho) = \langle x|\rho|x\rangle \, T(|x\rangle\langle x|) + \langle x^\perp|\rho|x^\perp\rangle \, T(|x^\perp\rangle\langle x^\perp|),$$

where $x^\perp$ is orthogonal to x in H. Moreover, if $T(|y\rangle\langle y|)$ is also a vertex of $\mathrm{Im}(T)$, different from $T(|x\rangle\langle x|)$, then $x \perp y$.

Part II

Characterization of universally image-additive channels

The (minimum output) von Neumann entropy

- The Shannon entropy of a probability vector $p = (p_i) \in \Delta_n$

$$H(p) = -\sum_{i=1}^{n} p_i \log p_i$$

- The von Neumann entropy of a quantum state $\rho \in \mathcal{S}_n$

$$H(\rho) = -\mathrm{Tr}(\rho \log \rho) = -\sum_{i=1}^{n} \lambda_i(\rho) \log \lambda_i(\rho)$$

- The entropy is additive: $H(\rho_1 \otimes \rho_2) = H(\rho_1) + H(\rho_2)$
- The minimum output entropy of a quantum channel T is

$$H_{\min}(T) = \min_{\rho \in \mathcal{S}_d} H(T(\rho))$$

Conjecture [Amosov, Holevo and Werner '00]

The quantity $H_{\min}$ is additive: for any quantum channels S, T

$$H_{\min}(S \otimes T) = H_{\min}(S) + H_{\min}(T)$$

Additivity of the minimum output entropy

Definition

A pair of channels (S, T) is called additive if $H_{\min}(S \otimes T) = H_{\min}(S) + H_{\min}(T)$. A channel T is called universally additive if the pair (S, T) is additive, for all channels S.

- Additivity of $H_{\min}$ (for all S, T) implies a simple formula for the capacity of channels to transmit classical information; in particular, it implies the additivity of the classical capacity
- The $\leq$ direction of the equality is trivial, take $\rho_{12} = \rho_1 \otimes \rho_2$
- Many channels are universally additive:
  - unitary: $T(\rho) = U\rho U^*$; in particular, the identity channel id
  - unital qubit: $T : \mathcal{M}_2 \to \mathcal{M}_2$, $T(I) = I$
  - depolarizing: $T(\rho) = (1-\lambda)\rho + \lambda I/d$
  - entanglement breaking
- But... the Additivity Conjecture is false ! [Hayden, Winter '08, Hastings '09]
- Counterexamples: if T is a random channel, then, with high probability, $(T, \bar{T})$ is non-additive

Image-additivity

Open question

Characterize universally additive quantum channels.

It is probably a hard question: the identity and entanglement breaking channels are "very different" in nature...

Definition

A pair of channels (S, T) is called image-additive if one of the following equivalent statements is satisfied:

1. The image of $S \otimes T$ is the convex hull of the tensor product of the images of S, T:
$$\mathrm{Im}(S \otimes T) = \mathrm{conv}\left[\mathrm{Im}(S) \otimes \mathrm{Im}(T)\right]$$
2. For every unit vector $\psi \in \mathbb{C}^{d_S} \otimes \mathbb{C}^{d_T}$, there is a separable state $\rho_{sep} \in \mathcal{S}_{d_S d_T}$ such that
$$[S \otimes T](|\psi\rangle\langle\psi|) = [S \otimes T](\rho_{sep})$$

A channel $T : \mathcal{M}_d \to \mathcal{M}_n$ is called universally image additive if the pair (S, T) is image-additive, for all channels S.

Remark: Image-additivity is stronger than additivity

Main result (universally image-additive channels)

Theorem

Let $T : \mathcal{M}_d \to \mathcal{M}_n$ be a quantum channel. The following assertions are equivalent:

1. T is universally image additive
2. T and $\mathrm{id} : \mathcal{M}_d \to \mathcal{M}_d$ are image additive
3. There exists an entanglement breaking channel $S : \mathcal{M}_d \to \mathcal{M}_d$ such that $T = T \circ S$
4. There exists an essentially classical-quantum channel $S : \mathcal{M}_d \to \mathcal{M}_d$ such that $T = T \circ S$
5. T is essentially classical-quantum

Proof ideas: 2 $\implies$ 3

- Choose $|\Phi_+\rangle$ to be the maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$

$$|\Phi_+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle$$

- Using the hypothesis, there exists a separable state ρ_{sep} such that

$$[T \otimes \mathrm{id}](|\Phi_+\rangle\langle\Phi_+|) = [T \otimes \mathrm{id}](\rho_{sep}).$$

- Thanks to the Choi-Jamiołkowski isomorphism, there exists an entanglement breaking channel S such that

$$\rho_{sep} = [S \otimes \mathrm{id}](|\Phi_+\rangle\langle\Phi_+|)$$

- Hence

$$[T \otimes \mathrm{id}]|\Phi_+\rangle\langle\Phi_+|) = [T \circ S \otimes \mathrm{id}](|\Phi_+\rangle\langle\Phi_+|)$$

- Using again the Choi-Jamiołkowski isomorphism, $T = T \circ S$, with S entanglement breaking

Proof ideas: 3 $\implies$ 4

- Starting from $T = T \circ S$, we get, by recurrence, $T = T \circ S^n$ for all $n \geq 1$, and thus $T = T \circ S_\infty$, where

$$S_\infty = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} S^n$$

- The channel S_∞ satisfies $S_\infty = S \circ S_\infty = S_\infty \circ S = S_\infty^2$. Thus, S_∞ is a completely positive projection on its image, the set of fixed points of S

$$\mathcal{F}_S = \{X \in \mathcal{M}_d \,:\, S(X) = X\}$$

- Given a quantum channel $S : \mathcal{M}_d \to \mathcal{M}_d$, there exist quantum states $\sigma_1, \ldots, \sigma_k \in \mathcal{S}_d$ having orthogonal supports such that

$$\mathcal{F}_S = 0_{V_T^\perp} \oplus \bigoplus_{i=1}^{k} \mathcal{M}_{d_i} \otimes \sigma_i$$

- Moreover, if S is entanglement breaking, the d_i above are all equal to 1, and thus the set of fixed points of T is spanned by density matrices $\sigma_1, \ldots, \sigma_k$ with orthogonal supports

$$\mathcal{F}_S = \mathrm{span}\{\sigma_1, \ldots, \sigma_k\}$$

and the channel S_∞ that projects on $\mathcal{F}_S$ is essentially classical-quantum

Thank you !

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