Bipartite unitary operators inducing unitarily invariant classes of quantum channels

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joint work with Julien Deschamps and Clément Pellegrini

Genova, July 1st 2015

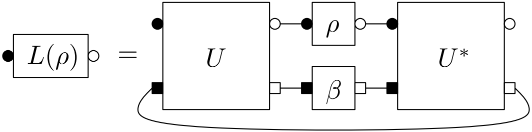
Stinespring dilation picture for quantum channels

Theorem

Any quantum channel $L : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ *(i.e. completely positive, trace preserving linear map) can be written as*

$$L(\rho) = [\mathrm{id} \otimes \mathrm{Tr}] \left(U(\rho \otimes \beta) U^* \right)$$

for some environment of size k *(*$k = n^2$ *suffices), a quantum state* $\beta \in \mathcal{M}_n^{1,+}(\mathbb{C})$ *and a global unitary operator* $U \in \mathcal{U}_{nk}$*.*



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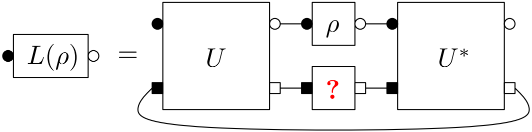
- What if we do not know / have access to β, the state of the environment ?



The main problem

$$L_{U,\beta}(\rho) := [\mathrm{id} \otimes \mathrm{Tr}]\left(U(\rho \otimes \beta)U^*\right)$$

Our mantra

Given a family $\mathcal{L}$ of quantum channels, characterize the set

$$\mathcal{U}_{\mathcal{L}} := \{U \in \mathcal{U}_{nk} \ : \ \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}), L_{U,\beta} \in \mathcal{L}\}.$$

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- If the set $\mathcal{L}$ is unitarily invariant, i.e.

$$L \in \mathcal{L} \iff \forall V_{1,2} \in \mathcal{U}_n,\ V_1 L(V_2 \cdot V_2^*) V_1^* \in \mathcal{L},$$

then the set $\mathcal{U}_{\mathcal{L}}$ is invariant by local unitary multiplication:

$$U \in \mathcal{U}_{\mathcal{L}} \iff \forall V_{1,2} \in \mathcal{U}_n, \forall W_{1,2} \in \mathcal{U}_k,\ (V_1 \otimes W_2) U (V_2 \otimes W_2) \in \mathcal{U}_{\mathcal{L}}.$$

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1. $\mathcal{L}_{aut} = \{V \cdot V^*\}_{V \in \mathcal{U}_n}$
2. $\mathcal{L}_{const} = \{\text{constant channels}\}$
3. $\mathcal{L}_{unital} = \{L \,:\, L(I) = I\}$
4. $\mathcal{L}_{mixed} = \mathrm{conv}\{V \cdot V^*\}_{V \in \mathcal{U}_n}$
5. $\mathcal{L}_{PPT} = \{L \,:\, L \text{ is PPT}\}$
6. $\mathcal{L}_{EB} = \{L \,:\, L \text{ is entanglement breaking}\}$

Unitary conjugations

$$\mathcal{U}_{aut} := \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}), L_{U,\beta}(\rho) = V_\beta \rho V_\beta^*\}.$$

Theorem

We have $\mathcal{U}_{aut} = \{V \otimes W \ : \ V \in \mathcal{U}_n, \ W \in \mathcal{U}_k\}$.
For $U = V \otimes W, \ L_{U,\beta}(\rho) = V \rho V^*$.

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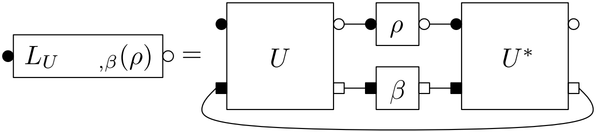
$$\mathcal{U}_{single} := \{U \in \mathcal{U}_{nk} \,|\, \text{the set } \{L_{U,\beta} \; : \; \beta \in \mathcal{M}_k^{1,+}(\mathbb{C})\} \text{ has 1 element}\}.$$

In other words, $U \in \mathcal{U}_{single}$ iff. the channel $L_{U,\beta}$ does not depend on β, the state of the environment.

Proposition

We have $\mathcal{U}_{single} = \mathcal{U}_{aut} = \{V \otimes W\}$.

Unitary conjugations



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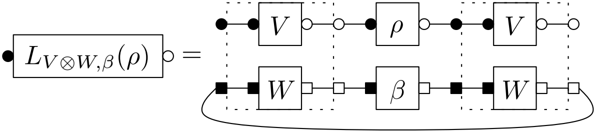

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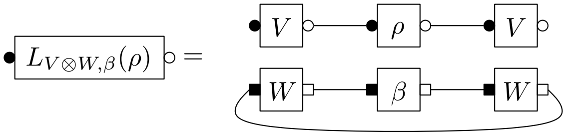

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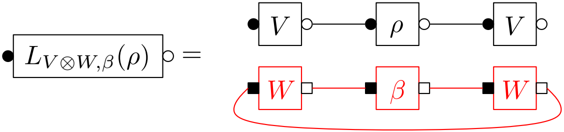

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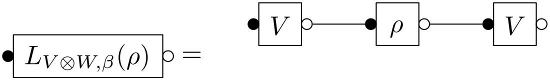

Constant channels

$$\mathcal{U}_{const} := \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}), L_{U,\beta} \text{ is a constant channel}\}.$$

Theorem

If $k \neq rn$ for $r = 1, 2, \ldots$, then $\mathcal{U}_{const}$ is empty. If $k = r \cdot n$ for some positive r, then

$$\mathcal{U}_{const} = \{(I_n \otimes V)(F_n \otimes I_r)(I_n \otimes W) \,:\, V, W \in \mathcal{U}_k\},$$

where $F_n \in \mathcal{U}_{n^2}$ denotes the flip operator.
For $U \in \mathcal{U}_{const}$ as above, $L_{U,\beta}(\rho) = [\mathrm{id}_n \otimes \mathrm{Tr}_r](W\beta W^*)$.

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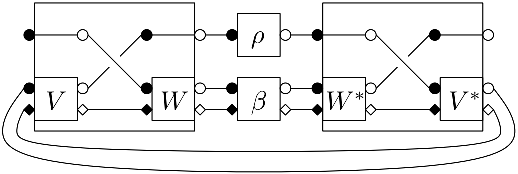
Corollary

If $n = k$, then $\mathcal{U}_{const} = F_n \cdot \mathcal{U}_{aut} = F_n \cdot \{V \otimes W \,:\, V, W \in \mathcal{U}_n\}$.

Constant channels



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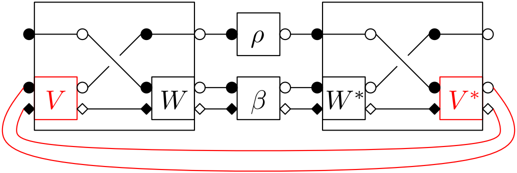

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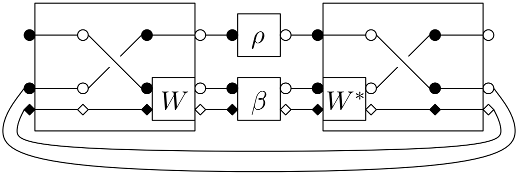

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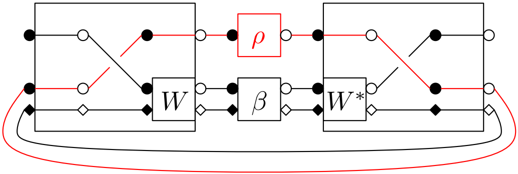

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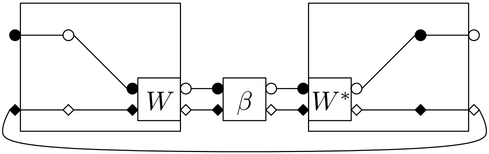

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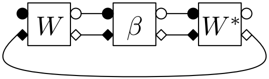

Unital channels

$$\mathcal{U}_{unital} := \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}), L_{U,\beta}(I) = I\}.$$

Theorem

One has

$$\mathcal{U}_{unital} = \mathcal{U}_{nk} \cap \mathcal{U}_{nk}^{\Gamma},$$

where $A^{\Gamma} = [\mathrm{id} \otimes \mathrm{transp}](A)$ *denotes the partial transposition of A. In other words,* $U \in \mathcal{U}_{unital}$ *iff. both U and* U^{Γ} *are unitary operators.*

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- $\mathcal{U}_{aut} = \{V \otimes W \, : \, V, W \in \mathcal{U}_n\} \subseteq \mathcal{U}_{unital}$.
- If $n > 1$, then $\mathcal{U}_{const} \cap \mathcal{U}_{unital} = \emptyset$.
- $\mathcal{U}_{unital}$ is a non-smooth algebraic variety, of dimension $nk(n + k - 1)$.

Block diagonal unitary matrices

Block-diagonal unitary operators with respect to the system A (resp. B)

$$\mathcal{U}^{A}_{block-diag} = \{U \in \mathcal{U}_{nk} \mid U = \sum_{i=1}^{k} U_i \otimes e_i f_i^*,$$

$$\text{with } U_i \in \mathcal{U}_n \text{ and } \{e_i\}, \{f_i\} \text{ orthonormal bases in } \mathbb{C}^k\}$$

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$$\mathcal{U}^{B}_{block-diag} = \{U \in \mathcal{U}_{nk} \,|\, U = \sum_{i=1}^{n} e_i f_i^* \otimes U_i,$$

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More generally, $U \in \mathcal{U}^A_{block-diag}$ iff.

$$U = \sum_{i=1}^{r} U_i \otimes R_i,$$

where U_i are unitary operators acting on $\mathbb{C}^n$ and R_i are partial isometries $R_i : \mathbb{C}^k \to \mathbb{C}^k$ such that $\sum_{i=1}^r R_i R_i^* = \sum_{i=1}^r R_i^* R_i = I_k$. Moreover, the decomposition is unique, up to the permutation of the terms in the sum and $\mathbb{C}U_i \neq \mathbb{C}U_j$ for $i \neq j$.

Block diagonal unitary matrices

Proposition

If $n = 2$*, then*

$$\mathcal{U}^{B}_{block-diag} \subseteq \mathcal{U}^{A}_{block-diag}.$$

In particular, when $n = k = 2$*, we have*

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In particular, when $n = k = 2$, we have

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$$\begin{aligned}
\mathcal{U}^{B}_{block-diag} \ni U &= e_1 f_1^* \otimes U_1 + e_2 f_2^* \otimes U_2 \\
&= (I \otimes U_1) \left[e_1 f_1^* \otimes I + e_2 f_2^* \otimes (U_1^* U_2) \right] \\
&= (I \otimes U_1) \left[e_1 f_1^* \otimes \left(\sum_{i=1}^{k} g_i g_i^* \right) + e_2 f_2^* \otimes \left(\sum_{i=1}^{k} \lambda_i g_i g_i^* \right) \right] \\
&= (I \otimes U_1) \sum_{i=1}^{k} (e_1 f_1^* + \lambda_i e_2 f_2^*) \otimes g_i g_i^* \\
&= (I \otimes U_1) \sum_{i=1}^{k} W_i \otimes g_i g_i^* = \sum_{i=1}^{k} W_i \otimes h_i g_i^* \in \mathcal{U}^{A}_{block-diag}.
\end{aligned}$$

Block diagonal unitary matrices

Proposition

A unitary operator U is block diagonal with respect to both tensor factors A and B (i.e. $U \in \mathcal{U}^A_{block-diag} \cap \mathcal{U}^B_{block-diag}$) iff.

$$U = \sum_{i=1}^{s} \sum_{j=1}^{r} \lambda_{ij} \; Q_i \otimes R_j \; ,$$

where, for all $i = 1, \ldots, s$, $j = 1, \ldots, r$, $|\lambda_{ij}| = 1$, and where $(Q_i)_{i=1,\ldots,s}$, $(R_j)_{j=1,\ldots,r}$ are two family of partial isometries respectively on $\mathbb{C}^n$ and $\mathbb{C}^k$ satisfying

$$\sum_{i=1}^{s} Q_i Q_i^* = \sum_{i=1}^{s} Q_i^* Q_i = I_n \quad , \quad \sum_{j=1}^{r} R_j R_j^* = \sum_{j=1}^{r} R_j^* R_j = I_k .$$

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- $\mathcal{U}^A_{block-diag}$ is a real algebraic variety of dimension $(n > 1)$ $\dim \mathcal{U}^A_{block-diag} = k(n^2 + 2k - 2)$.
- $\dim \mathcal{U}^A_{block-diag} \cap \mathcal{U}^B_{block-diag} = 2n^2 + 2k^2 + nk - 2n - 2k$.

Mixed quantum channels

$$\mathcal{U}_{mixed} := \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}),\ L_{U,\beta} \in \mathrm{conv}\{V \cdot V^*\}_{V \in \mathcal{U}_n}\}$$

$$= \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}),\ L_{U,\beta}(X) = \sum_{i=1}^{r(\beta)} p_i(\beta) U_i(\beta) X U_i(\beta)^*\}$$

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$$\text{with } p_i(\beta) \geq 0 \text{ and } \sum_i p_i(\beta) = 1\}$$

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We have the following chain of inclusions

$$\mathcal{U}^A_{block-diag} \subseteq \mathcal{U}_{prob-lin} \subseteq \mathcal{U}_{prob} \subseteq \mathcal{U}_{mixed} \subseteq \mathcal{U}_{unital}.$$

Mixed quantum channels

Theorem

We have $\mathcal{U}_{prob-lin} = \mathcal{U}^{A}_{block-diag}$.

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- Since $\beta \mapsto p_i(\beta)$ are linear, there exists a POVM (M_i) such that $p_i(\beta) = \mathrm{Tr}(M_i\beta)$.
- Prove the M_i's have orthogonal supports.
- Construct a candidate unitary operator $\tilde{U}$.
- Use the fact that

$$\forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}),\ L_{U,\beta} = L_{\tilde{U},\beta} \iff \exists W \in \mathcal{U}_k \text{ s.t. } U = (I_n \otimes W)\tilde{U}.$$

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Proposition

When $n = 2$, $\mathcal{U}^A_{block-diag} = \mathcal{U}_{unital}$, *so we have*

$$\mathcal{U}^A_{block-diag} = \mathcal{U}_{prob-lin} = \mathcal{U}_{prob} = \mathcal{U}_{mixed} = \mathcal{U}_{unital}.$$

Some work in progress...

Question

Characterize the unitarily invariant sets

$$\mathcal{U}_{PPT} = \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}), \quad L_{U,\beta} \text{ is a PPT channel } \}$$

$$\mathcal{U}_{EB} = \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}), \quad L_{U,\beta} \text{ is an entanglement breaking channel } \}.$$

Obviously, $\mathcal{U}_{const} \subseteq \mathcal{U}_{PPT} \subseteq \mathcal{U}_{EB}$. *Is there equality ?*

The End

thank you for your attention