Random quantum channels and additivity violations

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Herstmonceux, July 15th 2015

Outline of the talk

1. Random quantum channels and their minimum output entropy
2. Lower bounding $H^{\min}(\Phi \otimes \bar{\Phi})$
3. Computing $H^{\min}(\Phi)$
4. Additivity violations

Random quantum channels
and their minimum output entropy

Additivity for MOE of quantum channels

- Quantum channels: CPTP maps $\Phi : \mathcal{M}_{\text{in}}(\mathbb{C}) \to \mathcal{M}_{\text{out}}(\mathbb{C})$.
- Rényi entropies

$$p > 0 \quad H^p(\rho) = \frac{\log \mathrm{Tr} \rho^p}{1-p}, \quad H^1(\rho) = H(\rho) = -\mathrm{Tr}(\rho \log \rho).$$

- p-Minimal Output Entropy of a quantum channel

$$\begin{aligned} H^p_{\min}(\Phi) &= \min_{\rho \in \mathcal{M}^{1,+}_{\text{in}}(\mathbb{C})} H^p(\Phi(\rho)) \\ &= \min_{x \in \mathbb{C}^{\text{in}}} H^p(\Phi(P_x)). \end{aligned}$$

- Is the p-MOE additive ?

$$H^p_{\min}(\Phi \otimes \Psi) = H^p_{\min}(\Phi) + H^p_{\min}(\Psi) \quad \forall \Phi, \Psi.$$

- NO !!!
  - $p > 1$: Hayden + Winter '08;
  - $p = 1$: Hastings '08

Importance of additivity

- Simple formula for the (classical) capacity of quantum channels: if additivity holds, then there is no need to use inputs entangled over multiple uses of Φ.
- Shor '04 equivalence of additivity questions
  1. additivity of MOE
  2. additivity of the Holevo capacity χ ($= C_{\otimes}$ in Andreas' talk)
  3. (strong super-) additivity of the entanglement of formation E_F.
- Additivity proved for some particular channels: unital qubit, depolarizing, entanglement breaking, etc.
- Holevo-Werner channel violates additivity of the p-Rényi entropy for $p > 4.79$. No known deterministic examples for $p = 1$ of p close to 1.
- Difficult, mathematically challenging problem.

Random quantum channels

- Counterexamples to additivity conjectures are random.
- Random quantum channels from random isometries

$$\Phi(\rho) = \mathrm{Tr}_{\mathsf{anc}}(V\rho V^*),$$

where V is a Haar partial isometry

$$V : \mathbb{C}^{\mathsf{in}} \to \mathbb{C}^{\mathsf{out}} \otimes \mathbb{C}^{\mathsf{anc}}.$$

Equivalently, via the Stinespring dilation theorem

$$\Phi(\rho) = \mathrm{Tr}_{\mathsf{anc}}(U(\rho \otimes P_y)U^*),$$

where $y \in \mathbb{C}^{\frac{\mathsf{out}\times\mathsf{anc}}{\mathsf{in}}}$ and $U \in \mathcal{M}_{\mathsf{out}\times\mathsf{anc}}(\mathbb{C})$ is a Haar unitary matrix.
- Random quantum channels from i.i.d. random unitary matrices (random mixed unitary channels)

$$\Phi(\rho) = \sum_{i=1}^{k} p_i U_i \rho U_i^*,$$

for (random) probabilities p_i and i.i.d. Haar distributed unitary operators U_i.

Model of interest

Here, we focus on random quantum channels coming from random isometries, with the following parameters.

- in $= tnk$,
- out $= k$,
- anc $= n$,

where $n, k \in \mathbb{N}$ and $t \in (0,1)$. In general, we shall assume that

- $n \to \infty$
- k is fixed
- t is fixed.

In other words, we are interested in $\Phi : \mathcal{M}_{tnk}(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$,

$$\Phi(\rho) = [\mathrm{id}_k \otimes \mathrm{Tr}_n](V\rho V^*),$$

where V is a random isometry obtained by keeping the first tnk columns of a $nk \times nk$ Haar random unitary.

How to get counterexamples ?

- Choose Φ to be random and $\Psi = \bar{\Phi}$; this way, $H^p_{\min}(\Psi) = H^p_{\min}(\Phi)$.
- Bound

$$H^p_{\min}(\Phi \otimes \bar{\Phi}) \leq B_2 < 2B_1 \leq 2H^p_{\min}(\Phi).$$

Lower bounding $H^{\min}(\Phi \otimes \bar{\Phi})$

Strategy for B_2

- Remember: we want

$$H^p_{\min}(\Phi \otimes \bar{\Phi}) \leq B_2 < 2B_1 \leq 2H^p_{\min}(\Phi).$$

- Use trivial bound $H^p_{\min}(\Phi \otimes \bar{\Phi}) \leq H^p\left([\Phi \otimes \bar{\Phi}](X_{12})\right)$, for a particular choice of $X_{12} \in \mathcal{M}_{tnk}(\mathbb{C}) \otimes \mathcal{M}_{tnk}(\mathbb{C})$.
- $X_{12} = X_1 \otimes X_2$ do not yield counterexamples $\Rightarrow$ choose a maximally entangled state

$$X_{12} = E_{tnk} = \left(\frac{1}{\sqrt{tnk}} \sum_{i=1}^{tnk} e_i \otimes e_i\right) \left(\frac{1}{\sqrt{tnk}} \sum_{j=1}^{tnk} e_j \otimes e_j\right)^* .$$

- Bound entropies of the (random) density matrix

$$Z_n = [\Phi \otimes \overline{\Phi}](E_{tnk}) \in \mathcal{M}_k(\mathbb{C}) \otimes \mathcal{M}_k(\mathbb{C}).$$

Main result - finite rank output

Theorem (Collins + N. '09)

For all k, t, almost surely as $n \to \infty$, the eigenvalues of $Z_n = [\Phi \otimes \bar{\Phi}](E_{tnk})$ converge to

$$\left(t + \frac{1-t}{k^2}, \underbrace{\frac{1-t}{k^2}, \ldots, \frac{1-t}{k^2}}_{k^2-1 \text{ times}} \right) \in \Delta_{k^2}.$$

- Previously known bound (deterministic, comes from linear algebra): for all t, n, k, the largest eigenvalue of Z_n is at least t.
- Two improvements:
    1. "better" largest eigenvalue,
    2. knowledge of the whole spectrum.
- Precise knowledge of eigenvalues $\rightsquigarrow$ optimal estimates for entropies.
- However, smaller eigenvalues are the "worst possible".

Proof strategy for a.s. spectrum Z_n

- Use the method of moments
  1. Convergence in moments:

$$\mathbb{E}\mathrm{Tr}(Z_n^p) \to \left(t + \frac{1-t}{k^2}\right)^p + (k^2 - 1)\left(\frac{1-t}{k^2}\right)^p;$$

  2. Borel-Cantelli for a.s. convergence:

$$\sum_{n=1}^{\infty} \mathbb{E}\left[\left(\mathrm{Tr}(Z_n^p) - \mathbb{E}\mathrm{Tr}(Z_n^p)\right)^2\right] < \infty.$$

- We need to compute moments $\mathbb{E}\left[\mathrm{Tr}(Z_n^{p_1})^{q_1} \cdots \mathrm{Tr}(Z_n^{p_s})^{q_s}\right]$.
- Use the Weingarten formula to compute the unitary averages.

Unitary integration - Weingarten formula

- Using matrix coordinates, we can reduce our problem to computing integrals over the unitary group.

Theorem (Weingarten formula)

Let d be a positive integer and $(i_1, \ldots, i_p)$, $(i'_1, \ldots, i'_p)$, $(j_1, \ldots, j_p)$, $(j'_1, \ldots, j'_p)$ be p-tuples of positive integers from $\{1, 2, \ldots, d\}$. Then

$$\int_{\mathcal{U}(d)} U_{i_1 j_1} \cdots U_{i_p j_p} \overline{U_{i'_1 j'_1}} \cdots \overline{U_{i'_p j'_p}} \, dU = \sum_{\alpha, \beta \in \mathcal{S}_p} \delta_{i_1 i'_{\alpha(1)}} \ldots \delta_{i_p i'_{\alpha(p)}} \delta_{j_1 j'_{\beta(1)}} \ldots \delta_{j_p j'_{\beta(p)}} \mathrm{Wg}(d, \alpha\beta^{-1}).$$

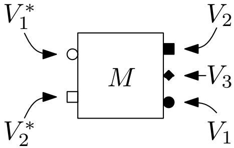
If $p \neq p'$ then

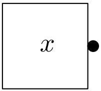
$$\int_{\mathcal{U}(d)} U_{i_1 j_1} \cdots U_{i_p j_p} \overline{U_{i'_1 j'_1}} \cdots \overline{U_{i'_{p'} j'_{p'}}} \, dU = 0.$$

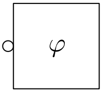
- There is a graphical way of reading this formula on the diagrams !

Boxes & wires

- Graphical formalism inspired by works of Penrose, Coecke, Jones, etc.
- Tensors $\rightsquigarrow$ decorated boxes.

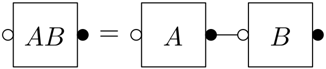


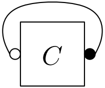




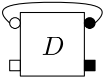
$M \in V_1 \otimes V_2 \otimes V_3 \otimes V_1^* \otimes V_2^*$ $\qquad$ $x \in V_1$ $\qquad$ $\varphi \in V_1^*$

- Tensor contractions (or traces) $V \otimes V^* \to \mathbb{C} \rightsquigarrow$ wires.



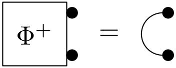


$\mathrm{Tr(C)}$



$\mathrm{Tr_{V_1}(D)}$

- Maximally entangled vector $\mathrm{Bell} = \sum_{i=1}^{\dim V} e_i \otimes e_i \in V \otimes V$

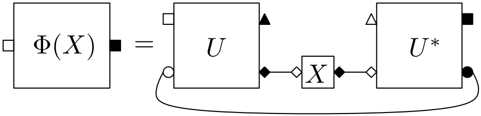


Graphical representation of quantum channels

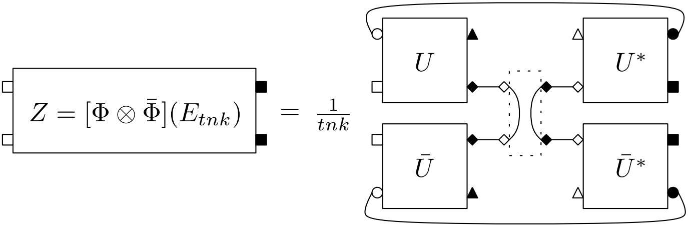
- Decorations/labels

$$\bullet\ \circ = \mathbf{C}^n \qquad \blacksquare\ \square = \mathbf{C}^k \qquad \blacklozenge\ \lozenge = \mathbf{C}^{tnk} \qquad \blacktriangle\ \triangle = \mathbf{C}^{t^{-1}}$$

- Single channel (finite rank output)



- Product of conjugate channels



“Graphical” Weingarten formula: graph expansion

Consider a diagram $\mathcal{D}$ containing random unitary matrices/boxes U and U^*. Apply the following removal procedure:

1. Start by replacing U^* boxed by $\overline{U}$ boxes (by reversing decoration shading).
2. By the (algebraic) Weingarten formula, if the number p of U boxes is different from the number of $\overline{U}$ boxes, then $\mathbb{E}\mathcal{D} = 0$.
3. Otherwise, choose a pair of permutations $(\alpha, \beta) \in \mathcal{S}_p^2$. These permutations will be used to pair decorations of $U/\overline{U}$ boxes.
4. For all $i = 1, \ldots, p$, add a wire between each white decoration of the i-th U box and the corresponding white decoration of the $\alpha(i)$-th $\overline{U}$ box. In a similar manner, use β to pair black decorations.
5. Erase all U and $\overline{U}$ boxes. The resulting diagram is denoted by $\mathcal{D}_{(\alpha,\beta)}$.

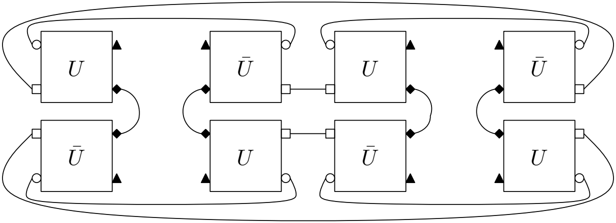
Theorem

$$\mathbb{E}\mathcal{D} = \sum_{\alpha,\beta \in \mathcal{S}_p} \mathcal{D}_{(\alpha,\beta)} \mathrm{Wg}(d, \alpha\beta^{-1}).$$

Example: $\mathbb{E}\mathrm{Tr}(Z^2)$

- We have to compute a sum over all pairings of 4 "U" boxes with 4 "$\overline{U}$" boxes.
- Diagrams associated to pairings are indexed by 2 permutations $(\alpha, \beta) \in \mathcal{S}_4^2$. Consider the permutation $\delta = (1\ 4)\ (2\ 3) \in \mathcal{S}_4$.

The original diagram



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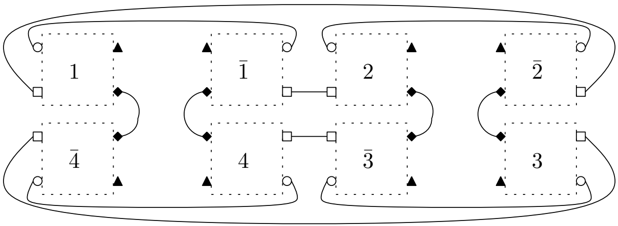
The diagram with the boxes removed



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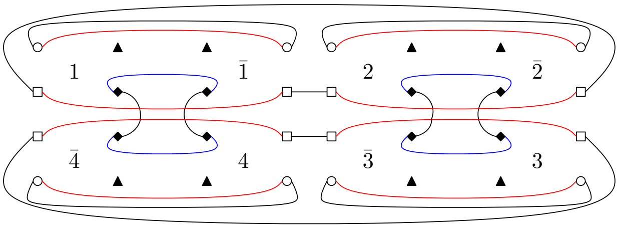
The wiring for $\alpha = \beta = \mathrm{id}$.



Contribution: $n^4 \cdot k^2 \cdot (tnk)^2 \cdot \mathrm{Wg}(\mathrm{id})$.

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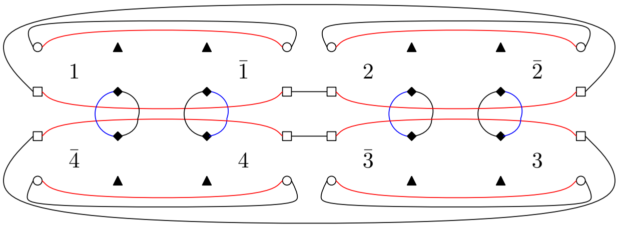

Contribution: $n^4 \cdot k^2 \cdot (tnk)^4 \cdot \mathrm{Wg}(\delta)$.

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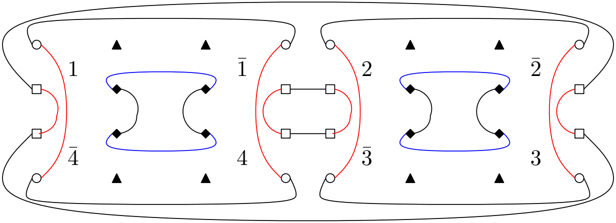

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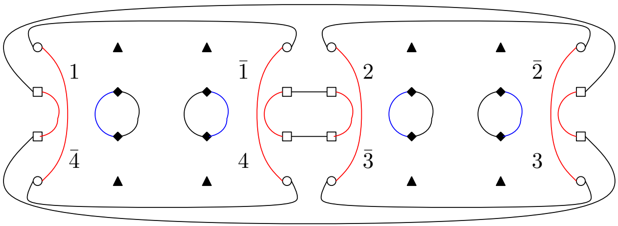


Contribution: $n^2 \cdot k^2 \cdot (tnk)^4 \cdot \mathrm{Wg}(\mathrm{id})$.

Sketch of the proof

- We want to compute, for all $p \geq 1$, $\mathbb{E}\mathrm{Tr}(Z^p)$.
- One needs to compute the contribution of each diagram $\mathcal{D}_{(\alpha,\beta)}$, where $\alpha, \beta \in \mathcal{S}_{2p}$.
- $\mathcal{D}_{(\alpha,\beta)}$ is a collection of loops associated to vector spaces of dimensions n, k, and tnk.
- Asymptotic for Weingarten weights ($\sigma \in \mathcal{S}_p$, $d \to \infty$, p fixed):

$$\mathrm{Wg}(d, \sigma) = d^{-(p+|\sigma|)}(\mathrm{Mob}(\sigma) + O(d^{-2})).$$

- One has to identify asymptotically dominating terms. Computations for fixed n are intractable due to the complexity of the Weingarten function. In the limit $n \to \infty$, the structure of the dominating terms is very simple.

Theorem (Collins + N. '09)

For all k, t, almost surely as $n \to \infty$,

$$\mathrm{spec}(Z_n) \to \left(t + \frac{1-t}{k^2}, \underbrace{\frac{1-t}{k^2}, \ldots, \frac{1-t}{k^2}}_{k^2-1 \text{ times}} \right) \in \Delta_{k^2}.$$

Computing $H^{\mathrm{min}}(\Phi)$

Strategy for B_1

- Remember: we want

$$H^p_{\min}(\Phi \otimes \bar{\Phi}) \leq B_2 < 2B_1 \leq 2H^p_{\min}(\Phi).$$

- We shall do more: we compute the exact limit (as $n \to \infty$) of $H^p_{\min}(\Phi)$.

Theorem (Belinschi, Collins, N. '13)

For all $p \geq 1$,

$$\lim_{n\to\infty} H_p^{min}(\Phi) = H_p(a, b, b, \ldots, b),$$

where a, b do not depend on p, $b = (1-a)/(k-1)$ and $a = \varphi(1/k, t)$ with

$$\varphi(s,t) = \begin{cases} s + t - 2st + 2\sqrt{st(1-s)(1-t)} & \text{if } s+t < 1; \\ 1 & \text{if } s+t \geq 1. \end{cases}$$

Entanglement of a vector

For a vector

$$x = \sum_{i=1}^{k} \sqrt{\lambda_i(x)} e_i \otimes f_i,$$

define $H(x) = H(\lambda(x)) = H(\rho) = -\sum_i \lambda_i(x) \log \lambda_i(x)$, the entropy of entanglement of the bipartite pure state x.

Note that

1. The state x is separable, $x = e \otimes f$, iff. $H(x) = 0$.
2. The state x is maximally entangled, $x = k^{-1/2} \sum_i e_i \otimes f_i$, iff. $H(x) = \log k$.

Recall that we are interested in computing

$$H^{\min}(\Phi) = \min_{x \in \mathbb{C}^d,\, \|x\|=1} H(\Phi(P_x)) = \min_{y \in \mathrm{Im} V,\, \|y\|=1} H([\mathrm{id}_k \otimes \mathrm{Tr}_n] P_y)$$

$$= \min_{y \in \mathrm{Im} V,\, \|y\|=1} H(y).$$

Entanglement of a subspace

For a subspace $V \subset \mathbb{C}^k \otimes \mathbb{C}^n$, define

$$\boxed{H_p^{\min}(V) = \min_{y \in V,\, \|y\|=1} H_p(y),}$$

the minimal entanglement of vectors in V.

Here, we abuse notation: recall that we are interested in random isometries $V : \mathbb{C}^{tnk} \to \mathbb{C}^k \otimes \mathbb{C}^n$. Since the quantities $H_p^{\min}$ only depend on the range of V, also write $V = \mathrm{ran} V$.

A subspace V is called entangled if $H^{\min}(V) > 0$, i.e. if it does not contain separable vectors $x \otimes y$.

Singular values of vectors from a subspace

$\rightsquigarrow$ Entropy is just a statistic, look at the set of all singular values directly!

For a subspace $V \subset \mathbb{C}^k \otimes \mathbb{C}^n$ of dimension $\dim V = d$, define the set eigen-/singular values or Schmidt coefficients

$$\boxed{K_V = \{\lambda(x) \,:\, x \in V, \|x\| = 1\}.}$$

$\rightsquigarrow$ Our goal is to understand K_V.

- The set K_V is a compact subset of the ordered probability simplex $\Delta_k^{\downarrow}$.
- Local invariance: $K_{(U_1 \otimes U_2)V} = K_V$, for unitary matrices $U_1 \in \mathcal{U}(k)$ and $U_2 \in \mathcal{U}(n)$.
- Monotonicity: if $V_1 \subset V_2$, then $K_{V_1} \subset K_{V_2}$.
- Recovering minimum entropies:

$$H_p^{\min}(\Phi) = H_p^{\min}(V) = \min_{\lambda \in K_V} H_p(\lambda).$$

Examples

The anti-symmetric subspace provides the (explicit) counter-example for the additivity of the p-Rényi entropy [Grudka, Horodecki, Pankowski '09].

- Let $k = n$ and put $V = \Lambda^2(\mathbb{C}^n)$
- The subspace V is almost half of the total space: $\dim V = n(n-1)/2$.
- Example of a vector in V:

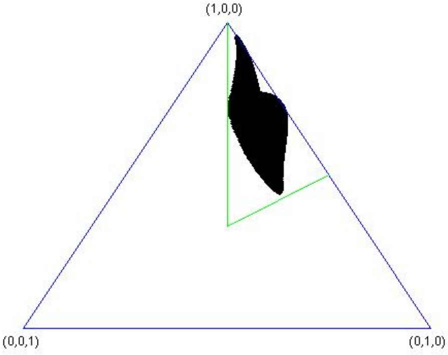
$$V \ni x = \frac{1}{\sqrt{2}}(e \otimes f - f \otimes e).$$

- Fact: singular values of vectors in V come in pairs.
- Hence, the least entropy vector in V is as above, with $e \perp f$ and $H(x) = \log 2$.
- Thus, $H^{\min}(V) = \log 2$ and one can show that

$$K_V = \{(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots) \in \Delta_n \, : \, \lambda_i \geq 0, \sum_i \lambda_i = 1/2\}.$$

Examples - K_V

$V = \mathrm{span}\{G_1, G_2\}$, where $G_{1,2}$ are 3×3 independent Ginibre random matrices.



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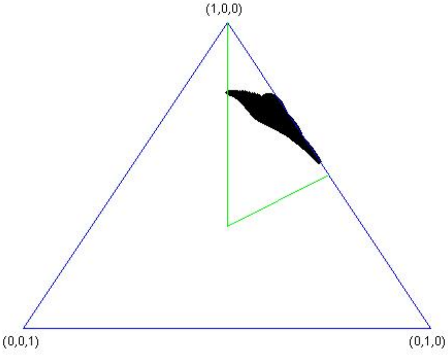

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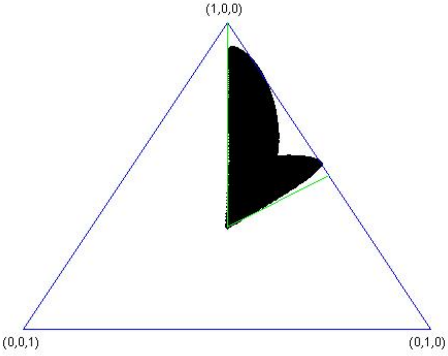
$V = \mathrm{span}\{\mathrm{I}_3, G\}$, where G is a 3×3 Ginibre random matrix.



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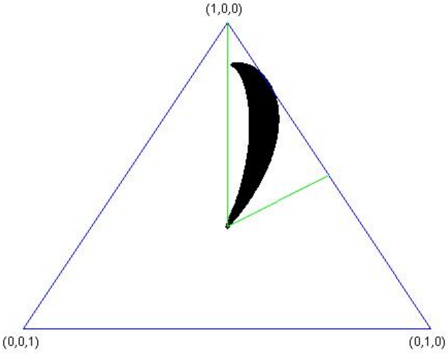

An open problem

Find explicit (i.e. non-random) examples of subspaces $V \subset \mathbb{C}^k \otimes \mathbb{C}^n$ with

1. large $\dim V$;
2. large $H^{\min}(V)$.

Main result

Recall that we are interested in random isometries/subspaces in the following asymptotic regime: k fixed, $n \to \infty$, and $d \sim tkn$, for a fixed parameter $t \in (0,1)$.

Theorem (Belinschi, Collins, N. '10)

For a sequence of uniformly distributed random subspaces V_n, the set K_{V_n} of singular values of unit vectors from V_n converges (almost surely, in the Hausdorff distance) to a deterministic, convex subset $K_{k,t}$ of the probability simplex Δ_k

$$K_{k,t} := \{\lambda \in \Delta_k \mid \forall x \in \Delta_k, \langle \lambda, x \rangle \leq \|x\|_{(t)}\}.$$

Corollary: exact limit of the minimum output entropy

By the previous theorem, in the specific asymptotic regime t, k fixed, $n \to \infty$, $d \sim tkn$, we have the following a.s. convergence result for random quantum channels Φ (defined via random isometries $V : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n$):

$$\lim_{n\to\infty} H_p^{\min}(\Phi) = \min_{\lambda \in K_{k,t}} H_p(\lambda).$$

It is not just a bound, the exact limit value is obtained.

Theorem (Belinschi, Collins, N. '13)

For all $p \geq 1$,

$$\lim_{n\to\infty} H_p^{min}(\Phi) = \min_{\lambda \in K_{k,t}} H_p(\lambda) = H_p(a, b, b, \ldots, b),$$

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$$\varphi(s,t) = \begin{cases} s + t - 2st + 2\sqrt{st(1-s)(1-t)} & \text{if } s + t < 1; \\ 1 & \text{if } s + t \geq 1. \end{cases}$$

Asymptotic freeness of random matrices

Theorem (Voiculescu '98)

Let (A_n) and (B_n) be sequences of $n \times n$ matrices such that A_n and B_n converge in distribution (with respect to $n^{-1}\mathrm{Tr}$) for $n \to \infty$. Furthermore, let (U_n) be a sequence of Haar unitary $n \times n$ random matrices. Then, A_n and $U_n B_n U_n^*$ are asymptotically free for $n \to \infty$.

If A_n, B_n are matrices of size n, whose spectra converge towards μ_a, μ_b, the spectrum of $A_n + U_n B_n U_n^*$ converges to $\mu_a \boxplus \mu_b$; here, $\mu_a \boxplus \mu_b$ is the distribution of $a + b$, where $a, b \in (\mathcal{A}, \tau)$ are free random variables having distributions resp. μ_a, μ_b.

If A_n, B_n are matrices of size n such that $A_n \geq 0$, whose spectra converge towards μ_a, μ_b, the spectrum of $A_n^{1/2} U_n B_n U_n^* A_n^{1/2}$ converges to $\mu_a \boxtimes \mu_b$.

Example: truncation of random matrices

Let $P_n \in \mathcal{M}_n$ a projection of rank $n/2$; its eigenvalues are 0 and 1, with multiplicity $n/2$. Hence, the distribution of P_n converges, when $n \to \infty$, to the Bernoulli probability measure $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

Let $C_n \in \mathcal{M}_{n/2}$ be the top $n/2 \times n/2$ corner of $U_n P_n U_n^*$, with U_n a Haar random unitary matrix. What is the distribution of C_n ?

Up to zero blocks, $C_n = Q_n(U_n P_n U_n^*)Q_n$, where Q_n is the diagonal orthogonal projection on the first $n/2$ coordinates of $\mathbb{C}^n$. The distribution of Q_n converges to $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

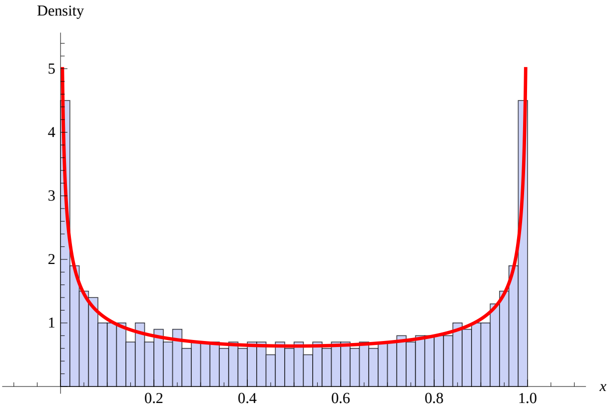
Free probability theory tells us that the distribution of C_n will converge to

$$(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1) \boxtimes (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1) = \frac{1}{\pi\sqrt{x(1-x)}}\mathbf{1}_{[0,1]}(x)dx,$$

which is the arcsine distribution.

Example: truncation of random matrices

Histogram of eigenvalues of a truncated randomly rotated projector of relative rank $1/2$ and size $n = 4000$; in red, the density of the arcsine distribution.



The t-norm

Definition

For a positive integer k, embed $\mathbb{R}^k$ as a self-adjoint real subalgebra $\mathcal{R}$ of a C^*-ncps $(\mathcal{A}, \tau)$, so that $\tau(x) = (x_1 + \cdots + x_k)/k$. Let p_t be a projection of rank $t \in (0, 1]$ in $\mathcal{A}$, free from $\mathcal{R}$. On the real vector space $\mathbb{R}^k$, we introduce the following norm, called the (t)-norm:

$$\|x\|_{(t)} := \|p_t x p_t\|_\infty,$$

where the vector $x \in \mathbb{R}^k$ is identified with its image in $\mathcal{R}$.

- One can show that $\|\cdot\|_{(t)}$ is indeed a norm, which is permutation invariant.
- When $t > 1 - 1/k$, $\|\cdot\|_{(t)} = \|\cdot\|_\infty$ on $\mathbb{R}^k$.
- $\lim_{t \to 0^+} \|x\|_{(t)} = k^{-1} |\sum_i x_i|$.

Corners of randomly rotated projections

Theorem (Collins '05)

In $\mathbb{C}^n$, choose at random according to the Haar measure two independent subspaces V_n and V'_n of respective dimensions $q_n \sim sn$ and $q'_n \sim tn$ where $s, t \in (0,1]$. Let P_n (resp. P'_n) be the orthogonal projection onto V_n (resp. V'_n). Then,

$$\lim_n \|P_n P'_n P_n\|_\infty = \varphi(s,t) = \sup \operatorname{supp}((1-s)\delta_0 + s\delta_1) \boxtimes ((1-t)\delta_0 + t\delta_1),$$

with

$$\varphi(s,t) = \begin{cases} s + t - 2st + 2\sqrt{st(1-s)(1-t)} & \text{if } s+t<1; \\ 1 & \text{if } s+t \geq 1. \end{cases}$$

Hence, we can compute

$$\| \underbrace{1, \cdots, 1}_{j \text{ times}}, \underbrace{0, \cdots, 0}_{k-j \text{ times}} \|_{(t)} = \varphi(\frac{j}{k}, t).$$

$K_{V_n} \to K_{k,t}$: idea of the proof

A simpler question: what is the largest maximal singular value $\max_{x\in V,\|x\|=1} \lambda_1(x)$ of vectors from the subspace V ?

$$\begin{aligned}
\max_{x\in V,\|x\|=1} \lambda_1(x) &= \max_{x\in V,\|x\|=1} \lambda_1([\mathrm{id}_k \otimes \mathrm{Tr}_n]P_x) \\
&= \max_{x\in V,\|x\|=1} \|[\mathrm{id}_k \otimes \mathrm{Tr}_n]P_x\| \\
&= \max_{x\in V,\|x\|=1} \max_{y\in\mathbb{C}^k,\|y\|=1} \mathrm{Tr}\left[([\mathrm{id}_k \otimes \mathrm{Tr}_n]P_x) \cdot P_y\right] \\
&= \max_{x\in V,\|x\|=1} \max_{y\in\mathbb{C}^k,\|y\|=1} \mathrm{Tr}\left[P_x \cdot P_y \otimes \mathrm{I}_n\right] \\
&= \max_{y\in\mathbb{C}^k,\|y\|=1} \max_{x\in V,\|x\|=1} \mathrm{Tr}\left[P_x \cdot P_y \otimes \mathrm{I}_n\right] \\
&= \max_{y\in\mathbb{C}^k,\|y\|=1} \|P_V \cdot P_y \otimes \mathrm{I}_n \cdot P_V\|_\infty .
\end{aligned}$$

The set $K_{k,t}$ and t-norms

- $K_{k,t} := \{\lambda \in \Delta_k \mid \forall x \in \Delta_k, \langle \lambda, x \rangle \le \|x\|_{(t)}\}$.
- Recall that

$$\max_{x \in V, \|x\|=1} \lambda_1(x) = \max_{y \in \mathbb{C}^k, \|y\|=1} \|P_V P_y \otimes \mathrm{I}_n P_V\|_\infty.$$

- For fixed y, P_V and $P_y \otimes \mathrm{I}_n$ are independent projectors of relative ranks t and $1/k$ respectively.
- Thus,

$$\|P_V \cdot P_y \otimes \mathrm{I}_n \cdot P_V\|_\infty \to \| ((1-t)\delta_0 + t\delta_1) \boxtimes ((1-1/k)\delta_0 + 1/k\delta_1) \| = \varphi(t, 1/k) = \|(1, 0, \ldots, 0)\|_{(t)}.$$

- We can take the max over y at no cost, by considering a finite net of y's, since k is fixed.
- To get the full result $\limsup_{n\to\infty} K_{V_n} \subset K_{k,t}$, use $\langle \lambda, x \rangle$ (for all directions x) instead of λ_1.
- The inclusion $\liminf_{n\to\infty} K_{V_n} \supset K_{k,t}$, is much easier, and follows from the convergence in distribution.

Additivity violations

Recall

$$H^p_{\min}(\Phi \otimes \bar\Phi) \leq B_2 < 2B_1 \leq 2H^p_{\min}(\Phi).$$

Theorem (Collins + N. '09)

For all k, t, almost surely as $n \to \infty$, if $Z_n = (\Phi \otimes \bar\Phi)(E_{tnk})$

$$\mathrm{spec}(Z_n) \to \left(t + \frac{1-t}{k^2}, \underbrace{\frac{1-t}{k^2}, \ldots, \frac{1-t}{k^2}}_{k^2-1 \ \textit{times}} \right) \in \Delta_{k^2}.$$

Theorem (Belinschi, Collins, N. '13)

For all $p \geq 1$,

$$\lim_{n\to\infty} H_p^{min}(\Phi) = H_p(a, b, b, \ldots, b),$$

where $b = (1-a)/(k-1)$ and $a = \varphi(1/k, t)$ with

$$\varphi(s,t) = \begin{cases} s + t - 2st + 2\sqrt{st(1-s)(1-t)} & \textit{if } s+t < 1; \\ 1 & \textit{if } s+t \geq 1. \end{cases}$$

Putting things together

Theorem (Belinschi, Collins, N. '13)

Using the limit for $H^{\min}(\Phi)$ and the upper bound for $H^{\min}(\Phi)$, the lowest dimension for which a violation of the additivity can be observed is $k = 183$. For large k, violations of size $1 - \varepsilon$ bits can be obtained.

How to improve this ?

1. Other asymptotic regimes
2. Use $\Psi \neq \bar{\Phi}$
3. For $\Phi \otimes \bar{\Phi}$, compute the actual limit of $H^{\min}(\Phi \otimes \bar{\Phi})$, and not just an upper bound.

The End

thank you for your attention