Bipartite unitary operators inducing special classes of quantum channels

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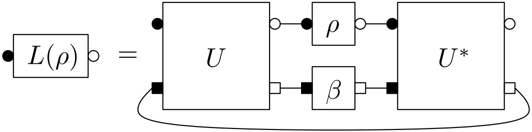
Stinespring dilation for quantum channels

Theorem

Any quantum channel $L : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ *(i.e. completely positive, trace preserving linear map) can be written as*

$$L(\rho) = [\mathrm{id} \otimes \mathrm{Tr}]\left(U(\rho \otimes \beta)U^*\right)$$

for some environment of size k *(*$k = n^2$ *suffices), a quantum state* $\beta \in \mathcal{M}_n^{1,+}(\mathbb{C})$ *and a global unitary operator* $U \in \mathcal{U}_{nk}$*.*



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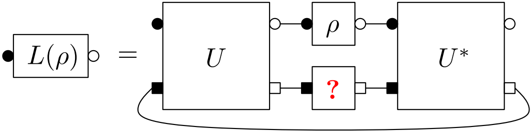
for some environment of size k ($k = n^2$ suffices), a quantum state $\beta \in \mathcal{M}_n^{1,+}(\mathbb{C})$ and a global unitary operator $U \in \mathcal{U}_{nk}$.

- What if we do not know / have access to β, the state of the environment ?



The main problem

$$L_{U,\beta}(\rho) := [\mathrm{id} \otimes \mathrm{Tr}]\left(U(\rho \otimes \beta)U^*\right)$$

Our mantra

Given a family $\mathcal{L}$ of quantum channels, characterize the set

$$\mathcal{U}_{\mathcal{L}} := \{U \in \mathcal{U}_{nk} \ : \ \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}), L_{U,\beta} \in \mathcal{L}\}.$$

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- If the set $\mathcal{L}$ is unitarily invariant, i.e.

  $$L \in \mathcal{L} \iff \forall V_{1,2} \in \mathcal{U}_n,\ V_1 L(V_2 \cdot V_2^*) V_1^* \in \mathcal{L},$$

  then the set $\mathcal{U}_{\mathcal{L}}$ is invariant by local unitary multiplication:

  $$U \in \mathcal{U}_{\mathcal{L}} \iff \forall V_{1,2} \in \mathcal{U}_n, \forall W_{1,2} \in \mathcal{U}_k,\ (V_1 \otimes W_2)U(V_2 \otimes W_2) \in \mathcal{U}_{\mathcal{L}}.$$

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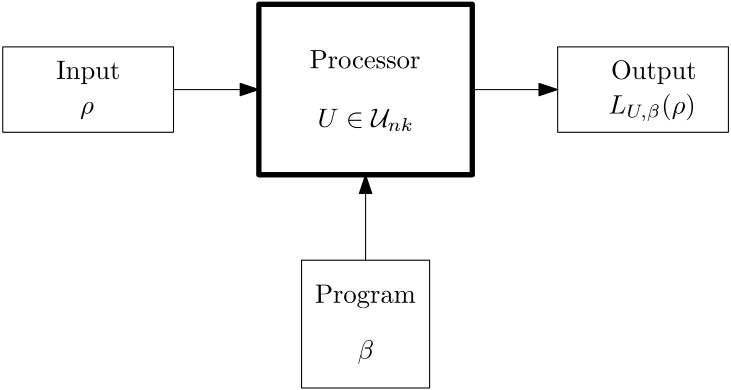
1. $\mathcal{L}_{aut} = \{V \cdot V^*\}_{V \in \mathcal{U}_n}$
2. $\mathcal{L}_{const} = \{\text{constant channels}\}$
3. $\mathcal{L}_{unital} = \{L \ : \ L(I) = I\}$
4. $\mathcal{L}_{mixed} = \mathrm{conv}\{V \cdot V^*\}_{V \in \mathcal{U}_n}$
5. $\mathcal{L}_{diag} = \{L \ : \ L(\mathrm{diag}) \subseteq \mathrm{diag}\}$
6. $\mathcal{L}_{tens} = \{L \ : \ L(\mathcal{M}_d(\mathbb{C}) \otimes I_r) \subseteq \mathcal{M}_d(\mathbb{C}) \otimes I_r\}$

Processor / program point of view

- Bužek, Ziman and collaborators study the same problem, under a different name



Lemma (Equivalent processors)

Two processors $U, V \in \mathcal{U}_{nk}$ are equivalent, i.e. for all programs $\beta \in \mathcal{M}_k^{1,+}(\mathbb{C})$, $L_{U,\beta} = L_{V,\beta}$, iff there exists $W \in \mathcal{U}_k$ s.t. $U = (I_n \otimes W)V$.

Unitary conjugations

$$\mathcal{U}_{aut} := \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}), L_{U,\beta}(\rho) = V_\beta \rho V_\beta^*\}$$

Theorem

We have $\mathcal{U}_{aut} = \{V \otimes W \,:\, V \in \mathcal{U}_n,\, W \in \mathcal{U}_k\}$.
For $U = V \otimes W,\ L_{U,\beta}(\rho) = V\rho V^*$.

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For $U = V \otimes W$, $L_{U,\beta}(\rho) = V \rho V^*$.

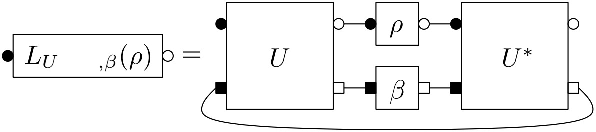
$$\mathcal{U}_{single} := \{U \in \mathcal{U}_{nk} \,|\, \text{the set } \{L_{U,\beta} \;:\; \beta \in \mathcal{M}_k^{1,+}(\mathbb{C})\} \text{ has 1 element}\}.$$

In other words, $U \in \mathcal{U}_{single}$ iff the channel $L_{U,\beta}$ does not depend on β, the state of the environment.

Proposition

We have $\mathcal{U}_{single} = \mathcal{U}_{aut} = \{V \otimes W\}$.

Unitary conjugations



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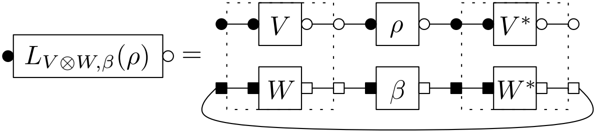

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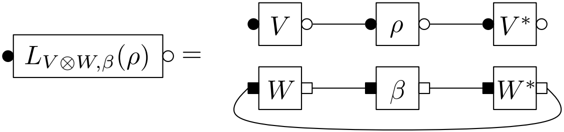

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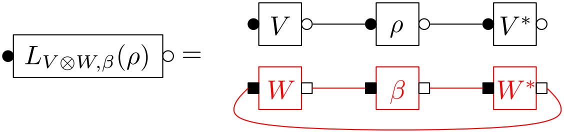

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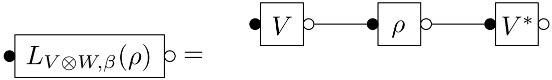

Constant channels

$$\mathcal{U}_{const} := \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}), L_{U,\beta} \text{ is a constant channel}\}.$$

Theorem

If $k \neq rn$ for $r = 1, 2, \ldots$, then $\mathcal{U}_{const}$ is empty. If $k = r \cdot n$ for some positive r, then

$$\mathcal{U}_{const} = \{(I_n \otimes V)(F_n \otimes I_r)(I_n \otimes W) \,:\, V, W \in \mathcal{U}_k\},$$

where $F_n \in \mathcal{U}_{n^2}$ denotes the flip operator.
For $U \in \mathcal{U}_{const}$ as above, $L_{U,\beta}(\rho) = [\mathrm{id}_n \otimes \mathrm{Tr}_r](W \beta W^*)$.

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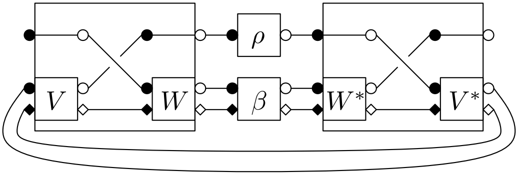
Corollary

If $n = k$, then $\mathcal{U}_{const} = F_n \cdot \mathcal{U}_{aut} = F_n \cdot \{V \otimes W \,:\, V, W \in \mathcal{U}_n\}$.

Constant channels



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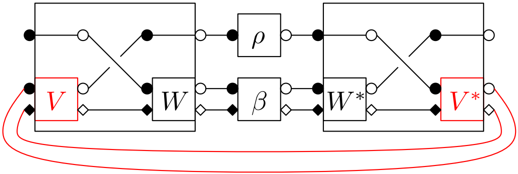

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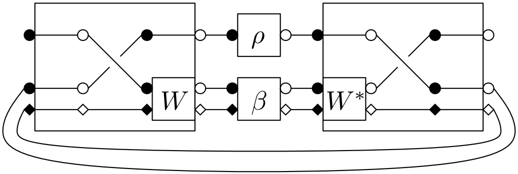

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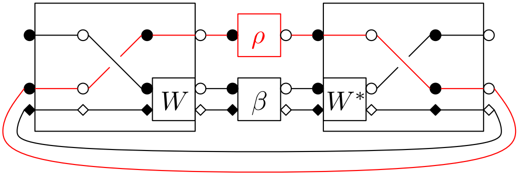

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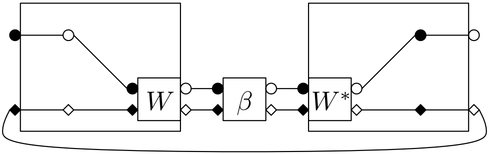

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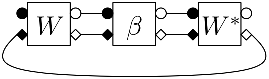

Unital channels

$$\mathcal{U}_{unital} := \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}), L_{U,\beta}(I) = I\}$$

Theorem

One has

$$\mathcal{U}_{unital} = \mathcal{U}_{nk} \cap \mathcal{U}_{nk}^{\Gamma}$$

where $A^{\Gamma} = [\mathrm{id} \otimes \mathrm{transp}](A)$ denotes the partial transposition of A. In other words, $U \in \mathcal{U}_{unital}$ iff both U and U^{Γ} are unitary operators.

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- $\mathcal{U}_{aut} = \{V \otimes W \,:\, V, W \in \mathcal{U}_n\} \subseteq \mathcal{U}_{unital}$.
- If $n > 1$, then $\mathcal{U}_{const} \cap \mathcal{U}_{unital} = \emptyset$.
- $\mathcal{U}_{unital}$ is a non-smooth algebraic variety. The dimension of the enveloping tangent space of $\mathcal{U}_{unital}$ is generically $nk(n + k - 1)$.

- Although it is easy to check whether a given U is an element of $\mathcal{U}_{unital}$, we do not know how to parametrize or to sample from $\mathcal{U}_{unital}$.
- We conjecture that the following algorithm produces (random) elements from $\mathcal{U}_{unital}$

Sampling from $\mathcal{U}_{unital}$

1. **Input:** Integers n, k and an error parameter $\varepsilon > 0$.
2. Start with a Haar distributed unitary random unitary operator $U \in \mathcal{U}_{nk}$.
3. While $\|U^\Gamma (U^\Gamma)^* - I_{nk}\|_2 > \varepsilon$, repeat the next step:
4. $U \leftarrow \mathrm{Pol}(U^\Gamma)$, where $\mathrm{Pol}(X)$ is the unitary operator V appearing in the polar decomposition of X: $X = VP$ with $P \geq 0$.
5. **Output:** U, an operator at distance at most ε from $\mathcal{U}_{unital}$.

Block diagonal unitary matrices

Block-diagonal (or control) unitary operators wrt the system A (resp. B)

$$\mathcal{U}^{A}_{block-diag} = \{U \in \mathcal{U}_{nk} \mid U = \sum_{i=1}^{k} U_i \otimes e_i f_i^*,$$

$$\text{with } U_i \in \mathcal{U}_n \text{ and } \{e_i\}, \{f_i\} \text{ orthonormal bases in } \mathbb{C}^k\}$$

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More generally, $U \in \mathcal{U}^{A}_{block-diag}$ iff

$$U = \sum_{i=1}^{r} U_i \otimes R_i,$$

where U_i are unitary operators acting on $\mathbb{C}^n$ and R_i are partial isometries $R_i : \mathbb{C}^k \to \mathbb{C}^k$ such that $\sum_{i=1}^{r} R_i R_i^* = \sum_{i=1}^{r} R_i^* R_i = I_k$. Moreover, the decomposition is unique, up to the permutation of the terms in the sum and $\mathbb{C}U_i \neq \mathbb{C}U_j$ for $i \neq j$.

Block diagonal unitary matrices

Proposition

If $n = 2$, then

$$\mathcal{U}^{B}_{block-diag} \subseteq \mathcal{U}^{A}_{block-diag}.$$

In particular, when $n = k = 2$, we have

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$$\begin{aligned}
\mathcal{U}^B_{block-diag} \ni U &= e_1 f_1^* \otimes U_1 + e_2 f_2^* \otimes U_2 \\
&= (I \otimes U_1)\left[e_1 f_1^* \otimes I + e_2 f_2^* \otimes (U_1^* U_2)\right] \\
&= (I \otimes U_1)\left[e_1 f_1^* \otimes \left(\sum_{i=1}^{k} g_i g_i^*\right) + e_2 f_2^* \otimes \left(\sum_{i=1}^{k} \lambda_i g_i g_i^*\right)\right] \\
&= (I \otimes U_1)\sum_{i=1}^{k}(e_1 f_1^* + \lambda_i e_2 f_2^*) \otimes g_i g_i^* \\
&= (I \otimes U_1)\sum_{i=1}^{k} W_i \otimes g_i g_i^* = \sum_{i=1}^{k} W_i \otimes h_i g_i^* \in \mathcal{U}^A_{block-diag}.
\end{aligned}$$

Mixed quantum channels

$$\mathcal{U}_{mixed} := \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}),\; L_{U,\beta} \in \mathrm{conv}\{V \cdot V^*\}_{V \in \mathcal{U}_n}\}$$

$$= \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}),\; L_{U,\beta}(X) = \sum_{i=1}^{r(\beta)} p_i(\beta) U_i(\beta) X U_i(\beta)^*\}$$

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$$\mathcal{U}_{prob} := \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}),\ L_{U,\beta}(X) = \sum_{i=1}^{r} p_i(\beta) U_i X U_i^*$$

$$\text{with } p_i(\beta) \geq 0 \text{ and } \sum_i p_i(\beta) = 1\}$$

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$$\mathcal{U}_{prob-lin} := \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}),\ L_{U,\beta}(X) = \sum_{i=1}^{r} p_i(\beta) U_i X U_i^*$$

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We have the following chain of inclusions

$$\mathcal{U}^A_{block-diag} \subseteq \mathcal{U}_{prob-lin} \subseteq \mathcal{U}_{prob} \subseteq \mathcal{U}_{mixed} \subseteq \mathcal{U}_{unital}.$$

Mixed quantum channels

Theorem

For all n, k*, we have* $\mathcal{U}_{prob-lin} = \mathcal{U}^{A}_{block-diag}$.

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For all n, k, we have $\mathcal{U}_{prob-lin} = \mathcal{U}^{A}_{block-diag}$.

- Since $\beta \mapsto p_i(\beta)$ are linear, there exists a POVM (M_i) such that $p_i(\beta) = \mathrm{Tr}(M_i \beta)$.
- Prove the M_i's have orthogonal supports.
- Construct a candidate unitary operator $\tilde{U}$.
- Use the lemma on equivalence of processors.

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Proposition

When $n = 2$, $\mathcal{U}^{A}_{block-diag} = \mathcal{U}_{unital}$, *so we have*

$$\mathcal{U}^{A}_{block-diag} = \mathcal{U}_{prob-lin} = \mathcal{U}_{prob} = \mathcal{U}_{mixed} = \mathcal{U}_{unital}.$$

Non-invariant structures

- We focus next on some classes of channels which depend on some particular choice of basis

$$\mathcal{L}_{diag} = \{L \,:\, L(\mathrm{diag}) \subseteq \mathrm{diag}\}$$
$$\mathcal{L}_{tens} = \{L \,:\, L(\mathcal{M}_d(\mathbb{C}) \otimes I_r) \subseteq \mathcal{M}_d(\mathbb{C}) \otimes I_r\}$$

- We shall study both Schrödinger (quantum channels) and Heisenberg (unital CP maps) pictures

$$L_{U,\beta}(\rho) = [\mathrm{id} \otimes \mathrm{Tr}]\left(U(\rho \otimes \beta)U^*\right)$$
$$T_{U,\beta}(\rho) = [\mathrm{id} \otimes \mathrm{Tr}]\left(U^*(\rho \otimes I_k)U(I_n \otimes \beta)\right)$$

- We write

$$\mathcal{U}_{S,diag} := \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}),\ L_{U,\beta} \in \mathcal{L}_{diag}\}$$
$$\mathcal{U}_{H,diag} := \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}),\ T_{U,\beta} \in \mathcal{L}_{diag}\}$$
$$\mathcal{U}_{S,tens} := \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}),\ L_{U,\beta} \in \mathcal{L}_{tens}\}$$
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Matrices of partial isometries

Definition

An operator $R \in M_{nk}(\mathbb{C}) \cong M_n(M_k(\mathbb{C}))$ is called a matrix of partial isometries if its blocks R_{ij} defined by $R = \sum_{i,j=1}^{n} e_i e_j^* \otimes R_{ij}$ are partial isometries. Let E_{ij} (resp. F_{ij}) be the initial (resp. final) spaces of the partial isometries R_{ij}. R is said to be of type (1,2,3,4) respectively if

1. For all $i \in [n]$, the subspaces $\{E_{ij}\}_{j \in [n]}$ form a partition of $\mathbb{C}^k$;
2. For all $i \in [n]$, the subspaces $\{F_{ij}\}_{j \in [n]}$ form a partition of $\mathbb{C}^k$;
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4. For all $j \in [n]$, the subspaces $\{F_{ij}\}_{i \in [n]}$ form a partition of $\mathbb{C}^k$.

Lemma

A matrix of partial isometries is unitary iff it is of type (2,3).

Strucutre preserving maps

Theorem

Let $\{e_1, \ldots, e_n\}$ be some fixed basis of $\mathbb{C}^n$ and let diag be the diagonal sub-algebra of $\mathcal{M}_n(\mathbb{C})$. We have

$$\begin{aligned}\mathcal{U}_{H,diag} &= \{\text{matrices of partial isometries of type } (2,3)\} \\ &= \{\text{unitary matrices of partial isometries}\} \\ \mathcal{U}_{S,diag} &= \{\text{matrices of partial isometries of type } (2,3,4)\}\end{aligned}$$

Theorem

In the case of the tensor product algebra, we have (here, $n = dr$)

$$\begin{aligned}\mathcal{U}_{H,tens} &= \{(I_d \otimes V) \cdot (W \otimes I_r), : \ V \in \mathcal{U}_{rk}, \ W \in \mathcal{U}_{dk}\} \\ \mathcal{U}_{S,tens} &= \{(I_d \otimes V^\Gamma) \cdot (W \otimes I_r), : \ V \in \mathcal{U}_{rk} \cap \mathcal{U}_{rk}^\Gamma, \ W \in \mathcal{U}_{dk}\}\end{aligned}$$

Quantum latin squares

- A matrix of partial isometries of type (1,2,3,4), with $n = k$ and $\dim E_{ij} = \dim F_{ij} = 1$ for all i, j, is called a quantum Latin square.

Definition

A quantum Latin square (QLS) of order n is a matrix $X = (x_{ij})_{i,j=1}^{n}$, where $x_{ij} \in \mathbb{C}^n$ are such that the vectors on each row (resp. column) of X form an orthonormal basis of $\mathbb{C}^n$.

- Each classical latin square L_{ij} and each orthonormal basis $\{e_i\}$ of $\mathbb{C}^n$ induces a QLS by setting $x_{ij} = e_{L_{ij}}$.
- There exist non-classical QLS [Musto, Vicary]:

$$X = \begin{bmatrix} |0\rangle & |1\rangle & |2\rangle & |3\rangle \\ \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) & \frac{1}{\sqrt{5}}(i|0\rangle + 2|3\rangle) & \frac{1}{\sqrt{5}}(2|0\rangle + i|3\rangle) & \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \\ \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) & \frac{1}{\sqrt{5}}(2|0\rangle + i|3\rangle) & \frac{1}{\sqrt{5}}(i|0\rangle + 2|3\rangle) & \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \\ |3\rangle & |2\rangle & |1\rangle & |0\rangle \end{bmatrix}$$

Sampling quantum Latin squares

- We conjecture that the following algorithm, which generalizes Sinkhorn's classical procedure, produces (random) quantum Latin squares

Non-commutative Sinkhorn algorithm for sampling QLS

1. **Input:** The dimension n and an error parameter $\varepsilon > 0$
2. Start with x_{ij} independent uniform points on the unit sphere of $\mathbb{C}^n$.
3. While X is not an ε-QLS, do the steps (4-6)
4. Define the matrix Y by making the rows of X unitary:
$$\forall i \in [n], \qquad y_{ij} = \mathrm{Pol}\left(\sum_{s=1}^{n} x_{is} e_s^* \right) \cdot e_j .$$
5. Define the matrix Z by making the columns of Y unitary:
$$\forall j \in [n], \qquad z_{ij} = \mathrm{Pol}\left(\sum_{s=1}^{n} y_{sj} e_s^* \right) \cdot e_i .$$
6. $X \leftarrow Z$.
7. **Output:** X, an ε-QLS.

Open questions / work in progress

Question (Mixed channels)

For all values of n, k, we conjecture that $\mathcal{U}^A_{block-diag} = \mathcal{U}_{mixed}$.

Question (Other sets of channels)

Characterize the unitarily invariant sets

$$\mathcal{U}_{PPT} = \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}),\ L_{U,\beta} \text{ is a PPT channel } \}$$

$$\mathcal{U}_{EB} = \{U \in \mathcal{U}_{nk} \,|\, \forall \beta \in \mathcal{M}_k^{1,+}(\mathbb{C}),\ L_{U,\beta} \text{ is an entanglement breaking channel } \}.$$

Obviously, $\mathcal{U}_{const} \subseteq \mathcal{U}_{EB} \subseteq \mathcal{U}_{PPT}$, *and, if* $n = k$, $\mathcal{U}^A_{block-diag} \cdot F_n \subseteq \mathcal{U}_{EB} \subseteq \mathcal{U}_{PPT}$. *Is there equality ?*

Question (Generating random bipartite unitary operators)

Show that the iterative algorithms for sampling from $\mathcal{U}_{unital}$ *and* $\mathcal{U}_{S,diag}$ *(or* $\mathcal{QLS}$*) converge, and study the distribution of the limit.*

The End

thank you for your attention

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- Musto, B., Vicary, J. *Quantum Latin squares and unitary error bases.* Preprint arXiv:1504.02715.