SOME ANALYTICAL ASPECTS OF HADAMARD MATRICES

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Contents

1. The Hadamard conjecture. Historical remarks

Historically, sign matrices with orthogonal rows (or columns) have been considered first by Sylvester in 1867 [Syl67], but the Hadamard conjecture has been introduced by Hadamard in 1893 [Had93]:

Conjecture 1.1. *For any dimension N which is a multiple of 4, there exists a $N \times N$* Hadamard matrix*, i.e. a sign matrix $H \in M_N(\pm 1)$ with orthogonal rows (or columns).*

In these notes, we would like to take the following perspective on the Hadamard problem. A Hadamard matrix belongs to two different sets: one hands, it is a (multiple of an) orthogonal matrix, hence, it has an *analytical flavor*. On the other hand, its entries are ± 1, hence it also has a *combinatorial flavor*. We call this setting the *Hadamard landscape*, see Figure 1. On the left hand side, we are in the domain of analysis, while on the right half of the picture, we find ourselves in combinatorics-land. We shall see that one can study Hadamard matrices looking from one half or the other half of the landscape. We shall discuss two such approaches: the historical one, based on determinants (Section 2), and a new approach, analytical in nature, based on the ℓ_1 norm (Section 4).

The smallest order for which no Hadamard matrix is known is $N = 668$. The most recent progress in this direction has been made by Kharaghani and Tayfeh-Rezaie in [KTR05], where they show that a Hadamard matrix of order 428 exists. In Figure 2, we have represented a Sylvester Hadamard matrix of order $2^9 = 512$ and the Hadamard matrix of order $N = 428$ found in [KTR05].

Several relaxation of the Hadamard conjecture have been considered. For example, if one is willing to consider *rectangular* matrices, the following parameter was introduced: for a given integer n, let $r(n)$ be the largest number r such that there exists a $r \times n$ ± 1-valued matrix H such that $HH^\top = nI_r$. The Hadamard conjecture is equivalent to the statement that $r(4k) = 4k$, for any integer k. de Launey and Gordon proved the following result in [dLG01].

Theorem 1.2. *Fix $\varepsilon > 0$. Then, assuming the Extended Riemann Hypothesis is true, for any $n = 4k$ large enough,*

$$r(n) \geq n/2 - n^{17/22+\varepsilon}.$$

In other words, for each allowed dimension, there exists "half" a Hadamard matrix.

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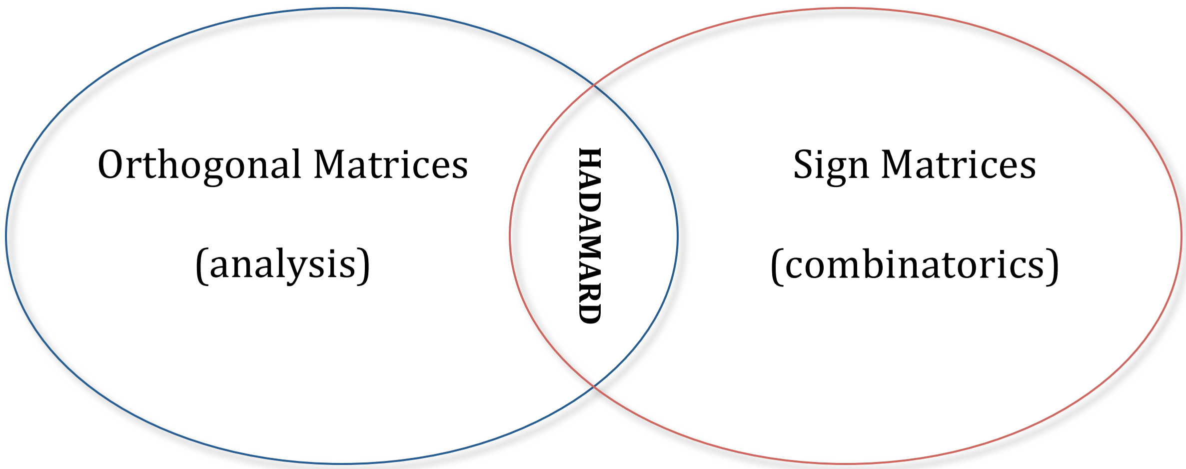

FIGURE 1. The Hadamard Landscape.

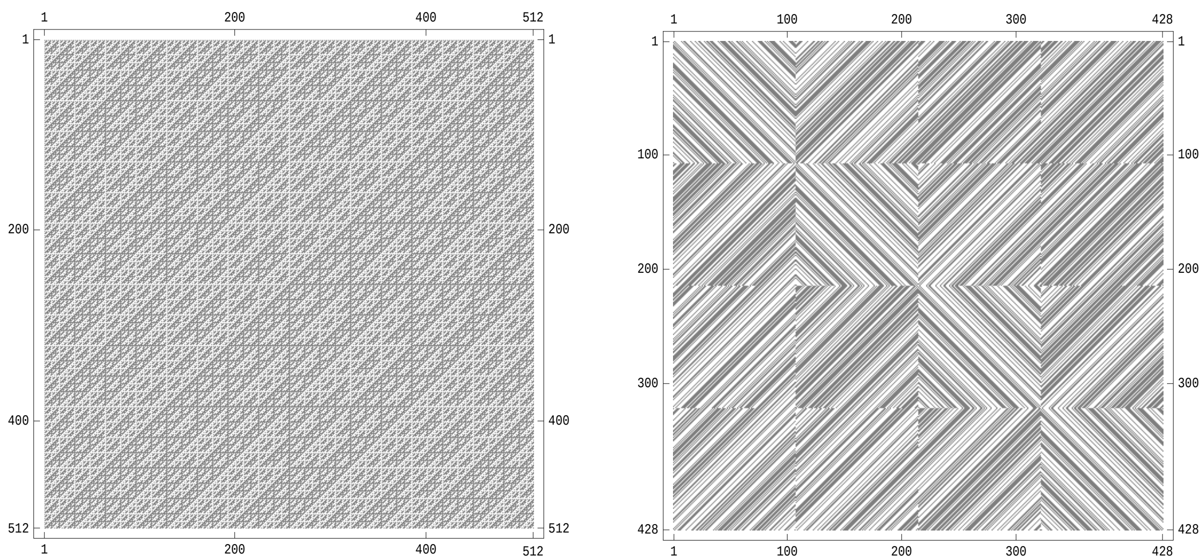

FIGURE 2. Hadamard matrices of order 512 (left) and 428 (right).

2. The determinant of sign matrices

Hadamard's initial approach in [Had93] came from the right hand side of the "Hadamard landscape": he considered the determinant function on ± 1 matrices.

Theorem 2.1. *Let X be a $N \times N$ complex matrix, with entries in the unit disk. Then,*

$$|\det X| \leq N^{N/2}.$$

A sign matrix X achieves equality iff X is Hadamard.

Proof. We shall use *Fischer's inequality* [Bha97, Problem II.5.6], which states that for any positive semidefinite matrix A and any pinching $\mathcal{C}$, $\det A \leq \det \mathcal{C}(A)$. We have

$$|\det X|^2 = \det(XX^*) \leq \prod_{i=1}^{N} (XX^*)_{ii} = \prod_{i=1}^{N} \|x_i\|^2 \leq N^N,$$

where we have denoted by x_i the i-th row of X. $\square$

Note that the determinant of sign matrices has some other interesting properties: it is easy to see that, for any sign matrix S of size N, 2^{N-1} divides $\det S$ (which is, of course, an integer); the values of p for which there exists a sign matrix S with $\det S = 2^{N-1}p$ have been recorded at http://www.indiana.edu/~maxdet/spectrum.html. Many inequalities exist for the maximum determinant of $N \times N$ sign matrices; see [OS07] for some recent progress, and sequence A003433 in [Slo16].

For random Bernoulli sign matrices (the entries S_{ij} of S are i.i.d. random variables, taking the values ± 1 with probability half), it has been shown [TV06] that the absolute value of the determinant of S is, with high probability, almost maximal (with respect to the exponent of N):

$$|\det S| \geq N^{(1/2-o(1))N}.$$

The probability that S is singular has also been thoroughly investigated. Important progress has been obtained recently in [BVW10], where it is shown that

$$\mathbb{P}(S \text{ is singular}) \leq (1/\sqrt{2} + o(1))^N,$$

improving on a constant of 3/4 obtained in [TV07].

The smallest and the largest value the permanent can take over sign matrices has also been considered, see [Wan05]. In the random case, Tao and Vu showed [TV09] that the permanent is of the same order as the determinant: almost surely as $N \to \infty$,

$$|\operatorname{per} S| \geq N^{(1/2-o(1))N}.$$

In both results, for the determinant and the permanent, one can replace the inequality by an equality, since the upper bound follows from Chebyshev's inequality and the following fact

$$\mathbb{E}\,|\det S|^2 = \mathbb{E}\,|\operatorname{per} S|^2 = N! = N^{(1+o(1))N}.$$

3. The ℓ_1 norm of orthogonal matrices

The concept of *almost Hadamard matrices* which will be discussed later was introduced in [BNŻ12], relying on an idea introduced in [BCS10]. The starting point is the following trivial observation. In the Hadamard landscape picture (1), Hadamard matrices lie at the intersection of two classes of matrices: (multiples of) orthogonal matrices and sign matrices. Note that both these sets of matrices have fixed Euclidean norm:

$$\begin{aligned} \forall\, U \in O(N), \qquad & \|U\|_2 = \sqrt{N} \\ \forall\, S \in M_N(\pm 1), \qquad & \|S\|_2 = N. \end{aligned}$$

We are interested in the intersection of these two classes of matrices, more precisely on deciding whether this intersection is empty or not. It is thus natural to consider the angle between these two sets.

Lemma 3.1. *Let, for a given dimension N,*

$$h_N := \max_{\substack{U \in O(N) \\ S \in M_N(\pm 1)}} \langle U, S \rangle = \max_{\substack{U \in O(N) \\ S \in M_N(\pm 1)}} \sum_{i,j=1}^N U_{ij} S_{ij} = \max_{U \in O(N)} \sum_{i,j=1}^N |U_{ij}|.$$

Consider the function $f : O(N) \to \mathbb{R}_+$ defined by

$$f(U) = \sum_{i,j=1}^N |U_{ij}|. \tag{1}$$

Then, for any orthogonal matrix U, $f(U) \leq N\sqrt{N}$, with equality iff $\sqrt{N}U$ is a Hadamard matrix. Equivalently,

$$h_N \leq N\sqrt{N},$$

with equality iff there exists a $N \times N$ Hadamard matrix.

Proof. Apply Cauchy-Schwarz:

$$\left(\sum_{i,j=1}^N |U_{ij}| \cdot 1 \right) \leq \sqrt{\sum_{i,j=1}^N |U_{ij}|^2} \cdot \sqrt{\sum_{i,j=1}^N 1^2} = N\sqrt{N}.$$

The equality case corresponds to $|U_{ij}| = \text{const}$, which is the case iff $|U_{ij}| = 1/\sqrt{N}$, i.e. iff $\sqrt{N}U$ is Hadamard. □

4. Almost Hadamard matrices

We have seen in the previous section that the global maxima of the "component-wise" 1-norm on the orthogonal group are precisely the Hadamard matrices. Taking the analyst's viewpoint, we consider the local maxima instead.

Definition 4.1. *An* almost Hadamard *matrix (AHM) is a square matrix $H \in M_N(\mathbb{R})$ such that $U = H/\sqrt{N} \in O(N)$ is a local maximum of the 1-norm on $O(N)$.*

One important feature of the above definition is that AHM exist for any dimension, not only multiples of 4. The function f from (1) is differentiable at points U having only non-zero elements. By a classical trick, we show next that we can restrict the search for local maxima only to such points (see [BCS10, Lemma 3.1] for a proof).

Lemma 4.2. *If a matrix U is a local maximum for the 1-norm on $O(N)$, then $U_{ij} \neq 0$ for all i, j.*

Let $O(N)^* = O(N) \cap M_N(\mathbb{R}^*)$ be the open set where f is differentiable (actually, piecewise linear). We have the following important result, the first half of which has been proven in [BCS10, BNŻ12].

Proposition 4.3. *Given a matrix $U \in O(N)^*$, write $S = \operatorname{sign}(U) \in M_N(\pm 1)$ for the sign matrix of U, that is $S_{ij} = \operatorname{sign}(U_{ij})$. Then, U is*

- *a critical point for f iff $U^\top S$ is a symmetric matrix*
- *a local maximizer for f iff the sum of the two smallest eigenvalues of $U^\top S$ is non-negative.*

Proof. Let us evaluate the function f on a path in the orthogonal group passing through U. Recall that the tangent space (at the identity) to the orthogonal group is the vector space of anti-symmetric matrices $A \in M_N(\mathbb{R})$, $A^\top = -A$. For such an A, consider the function F defined on a neighborhood of 0 by $F(t) = f(Ue^{tA})$. Since the perturbation t is small, we have, for all i, j, $|(Ue^{tA})_{ij}| = S_{ij}(Ue^{tA})_{ij}$. The first two derivatives of F read (again, for t small enough):

$$F'(t) = \sum_{i,j=1}^N S_{ij}(UAe^{tA})_{ij}$$

$$F''(t) = \sum_{i,j=1}^N S_{ij}(UA^2e^{tA})_{ij}.$$

Evaluating the derivatives at $t = 0$, we get

$$\begin{aligned}
F'(0) &= \sum_{i,j=1}^{N} S_{ij}(UA)_{ij} \\
&= \langle S, UA\rangle_{HS} \\
&= \langle U^\top S, A\rangle_{HS} \\
F''(0) &= \sum_{i,j=1}^{N} S_{ij}(UA^2 e^{tA})_{ij} \\
&= \langle S, UA^2\rangle_{HS} \\
&= \langle U^\top S, A^2\rangle_{HS}.
\end{aligned}$$

We now have $F'(0) = 0$ iff $U^\top S$ is orthogonal to all anti-symmetric matrices, i.e. $U^\top S$ is symmetric. The second claim follows from Lemma 4.4. □

Lemma 4.4. *Let $X \in M_N(\mathbb{R})$ be a symmetric operator. The following two conditions are equivalent:*

(1) The sum of the two smallest eigenvalues of X is non-negative
(2) For all anti-symmetric matrices A, $\langle X, A^2\rangle \leq 0$.

Proof. Let $a = \operatorname{vec} A$ be the *vectorization* of A, that is

$$a = \sum_{i,j=1}^{N} A_{ij} e_i \otimes e_j, \qquad \text{for} \qquad A = \sum_{i,j=1}^{N} A_{ij} e_i e_j^*.$$

Since A is an anti-symmetric matrix, the same is true for a: $a \in \Lambda^2(\mathbb{R}^N)$. It is clear (see Figure 3) that

$$\langle X, A^2\rangle = -\langle AX, A\rangle = -\langle a, (I_N \otimes X)a\rangle,$$

so the second point in the statement is equivalent to $P_-(I_N \otimes X)P_-$ being a PSD matrix, with

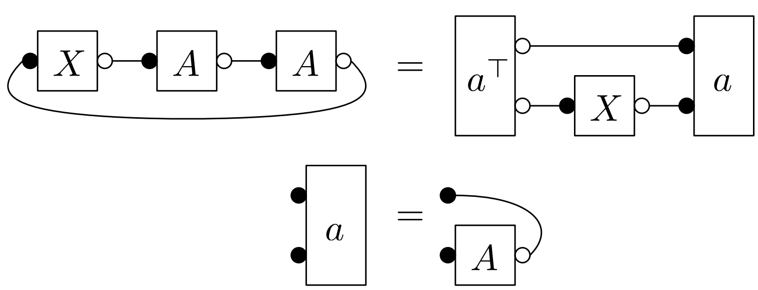

FIGURE 3. From anti-symmetric matrices to anti-symmetric vectors.

P_- being the orthogonal projector on the anti-symmetric subspace in $\mathbb{R}^N \otimes \mathbb{R}^N$. However, for any two orthogonal eigenvectors x, y of X having respective eigenvalues λ_x, λ_y, we have

$$P_-(I_N \otimes X)P_-(x \otimes y - y \otimes x) = P_-(\lambda_y x \otimes y - \lambda_x y \otimes x) = \frac{\lambda_x + \lambda_y}{2}(x \otimes y - y \otimes x),$$

showing that the non-trivial eigenvalues of $P_-(I_N \otimes X)P_-$ are $(\lambda_x + \lambda_y)/2$ for every ordered pair of distinct eigenvalues $\{\lambda_x, \lambda_y\}$ of X. □

The condition on the two smallest eigenvalues of the matrix $U^\top S$ in the result above seems non-intuitive. We show next a stronger condition is actually satisfied for critical points.

Conjecture 4.5. *Let $S \in M_N(\pm 1)$ be an (invertible) sign matrix, and consider its SVD $S = V\Delta W^T$ for some diagonal matrix with positive entries Δ and $V, W \in O(N)$. For any diagonal matrix $\Sigma \in M_N^{diag}(\pm 1)$, define $U_\Sigma := V\Sigma W^T$. If $S = \operatorname{sign}(U)$, then $\Sigma = I_N$ and thus $U = \operatorname{Pol}(S)$.*

If the previous conjecture holds, then for any (invertible) sign matrix S, there is at most one AHM having sign S: $U = \operatorname{Pol}(S)$. So one can enumerate AHM by computing, for any sign matrix S its polar part, and checking whether

$$S = \operatorname{sign}(\operatorname{Pol}(S)). \tag{2}$$

This suggests studying the properties of the (partially defined) dynamical system on sign matrices:

$$S \mapsto S' := \operatorname{sign}(\operatorname{Pol}(S)).$$

Let us introduce now some families of AHM, taken from [BNŻ12]. It is clear that Hadamard matrices are AHM. For any positive integer N, the matrix

$$K_N = \frac{1}{N}\begin{pmatrix} 2-N & 2 & \dots & 2 & 2 \\ 2 & 2-N & \dots & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & \dots & \dots & \dots & \dots \\ 2 & 2 & \dots & \dots & 2-N \end{pmatrix}$$

is unitary, and one checks easily it is almost Hadamard. These matrices are the most basic examples of AHM, and they exist for all dimensions.

For odd N, the following matrix is also AH:

$$L_N = \frac{1}{N}\begin{pmatrix} 1 & -\cos^{-1}\frac{\pi}{N} & \cos^{-1}\frac{2\pi}{N} & \dots & \cos^{-1}\frac{(N-1)\pi}{N} \\ \cos^{-1}\frac{(N-1)\pi}{N} & 1 & -\cos^{-1}\frac{\pi}{N} & \dots & -\cos^{-1}\frac{(N-2)\pi}{N} \\ \dots & \dots & \dots & \dots & \dots \\ -\cos^{-1}\frac{\pi}{N} & \cos^{-1}\frac{2\pi}{N} & -\cos^{-1}\frac{3\pi}{N} & \dots & 1 \end{pmatrix}.$$

Another family, defined only for N such that $N = q^2 + q + 1$, where $q = p^k$ is a prime power, comes from the adjacency matrix of the projective plane over $\mathbb{F}_q$. Here is for instance the matrix associated to the Fano plane ($q = 2$, see Figure 4), where $x = 2 - 4\sqrt{2}$, $y = 2 + 3\sqrt{2}$:

$$I_7 = \frac{1}{14}\begin{pmatrix} x & x & y & y & y & x & y \\ y & x & x & y & y & y & x \\ x & y & x & x & y & y & y \\ y & x & y & x & x & y & y \\ y & y & x & y & x & x & y \\ y & y & y & x & y & x & x \\ x & y & y & y & x & y & x \end{pmatrix}$$

Using the characterization of AHM in terms of sign matrices, one can exhaustively enumerate all AHM by simply checking condition (2) for all sign matrices. Picking the best local maxima for the function f, we could, with the help of a computer, show results in Table 4 for the quantities h_N and the matrices which achieve this maxima. Some of these values (as well as the values for larger N) were conjectured in [BNŻ12].

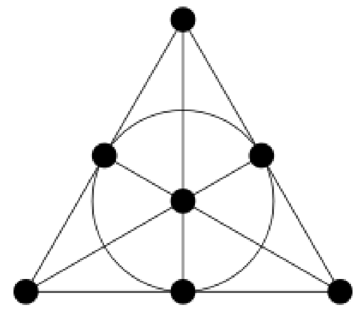

FIGURE 4. The Fano plane.

N	h_N	argmax f
2	$2\sqrt{2}$	H_2
3	5	K_3
4	8	H_4
5	11	K_5
6	$10\sqrt{2}$	$H_2 \otimes K_3$
7	$1+12\sqrt{2}$	I_7

TABLE 1. Values for the maximum ℓ_1 norm on $O(N)$. Blue values indicate the presence of a Hadamard matrix.

5. Submatrices of (almost) Hadamard matrices

Many of the known explicit constructions of new classes of Hadamard matrices use lower order Hadamard matrices as building blocks. The Sylvester matrix is an example of this kind of construction: if H_N is a $N \times N$ Hadamard matrix, then

$$\begin{bmatrix} H_N & H_N \\ H_N & -H_N \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes H_N$$

is also Hadamard. This motivates the study of submatrices of (almost) Hadamard matrices, see [Vij76, FRW88] Regarding almost Hadamard matrices / sign patterns, the following result has been proven in [BNS14].

Theorem 5.1. *Given a Hadamard matrix* $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_N(\pm 1)$ *with* $A \in M_r(\pm 1)$*,* D *is an almost Hadamard sign pattern (AHP) if:*

(1) A *is invertible, and* $r = 1, 2, 3$ *or* $N > \frac{r^2}{4}(r + \sqrt{r^2+8})^2$

(2) A *is Hadamard, and* $N > r(r-1)^2$.

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