Introduction à la théorie quantique de l'information

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Plan

1. Information et calcul quantiques
2. Vers une théorie de Shannon quantique: états et canaux
3. États quantiques aléatoires et la transposée partielle
4. Canaux quantiques aléatoires et la propriété d'additivité

Information et calcul quantiques

Théorie quantique de l'information

La théorie quantique de linformation est essentiellement divisée en deux secteurs:

1. le calcul quantique : les algorithmes quantiques
2. la la communication de linformation quantique : les protocoles de transmission (sécurisée) des données (quantiques).

La théorie quantique exploite les propriétés de la mécanique quantique, telles que

- la superposition : l'espace d'états d'un système quantique est linéaire. Dans la théorie classique, l'information est codée dans des bits, qui ne peuvent prendre que les valeurs discrètes 0 et 1. Au contraire, un qubit est un vecteur de norme 1 de l'espace $\mathbb{C}^2 = \text{span}\{|0\rangle, |1\rangle\}$.
- l'intrication : il existe des systèmes quantiques constitués de plusieurs composantes, dont l'état ne peut être décrit en termes des états des parties constituantes.

Protocoles et algorithmes quantiques

- 1982: Feynman propose d'utiliser un ordinateur quantique pour simuler des systèmes quantiques.
- 1984: Bennett et Brassard inventent un mécanisme d'échange de clé quantique [BB84], dont la sécurité repose sur une hypothèse physique.
- 1989: BB84 réalisé dans une expérience.
- 1992: Deutsch et Jozsa donne le premier example d'un algorithme quantique qui est plus efficace qu'un algorithme classique
- **1994**: Shor invente un algorithme quantique pour factoriser un entier naturel N en temps $O(\log^3 N)$ vs. $O(\exp(\log^{1/3} N))$ pour le meilleur algorithme classique connu.
- 2012: $21 = 3 \times 7$ experimentallement en utilisant des photons.
- 2015: D-Wave Systems, la première entreprise d'informatique quantique, annonce un ordinateur quantique (non-universel) avec 1000 qubits.
- 2018: La course vers la suprématie quantique: environ 50-100 qubits.

Vers une théorie de Shannon quantique: états et canaux

Quantum states

States	Deterministic	Random mixture
Classical	$x \in \{1, 2, \ldots, d\}$	$p \in \mathbb{R}^d$, $p_i \geq 0$, $\sum_i p_i = 1$
Quantum	$\psi \in \mathbb{C}^d$, $\|\psi\| = 1$	$\rho \in \mathcal{M}_d(\mathbb{C})$, $\rho \geq 0$, $\operatorname{Tr} \rho = 1$

- Quantum systems with d degrees of freedom are described by density matrices or mixed states

$$\rho \in \mathcal{M}^{1,+}(\mathbb{C}^d); \qquad \operatorname{Tr} \rho = 1 \text{ and } \rho \geq 0.$$

- Pure states are the particular case of rank one projectors, and correspond to unit vectors $\psi \in \mathbb{C}^d$

$$|\psi\rangle\langle\psi| \in \mathcal{M}^{1,+}(\mathbb{C}^d).$$

They are the extreme points of the convex body $\mathcal{M}^{1,+}(\mathbb{C}^d)$

Entanglement

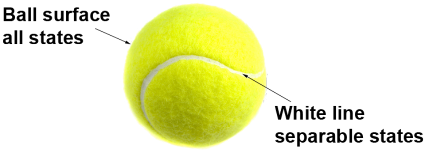
- Two quantum systems: $\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$.
- A mixed state ρ_{AB} is called separable if it can be written as a convex combination of product states

$$\rho_{AB} \in \mathcal{SEP} \iff \rho_{AB} = \sum_i t_i \sigma_i^{(A)} \otimes \sigma_i^{(B)},$$

with $t_i \geq 0$, $\sum_i t_i = 1$, $\sigma_i^{(A,B)} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_{A,B}})$.

- Non-separable states are called entangled.

- Pure states: $|x\rangle\langle x|$ is separable $\iff$ $x = y \otimes z$.
- All bi-partite quantum pure states have dimension $d_A d_B - 1$, whereas product states have dimension $d_A + d_B - 2$, which is strictly smaller $\implies$ a generic pure state is entangled!



Mixed state entanglement is hard, but...

- Deciding if a given ρ_{AB} is separable is NP-hard. Detecting entanglement for general states is a difficult, central problem in QIT.

- A linear map $f : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^{d'})$ is called
  - positive if $A \geq 0 \implies f(A) \geq 0$;
  - completely positive if $\mathrm{id}_k \otimes f$ is positive for all $k \geq 1$.

- If $f : \mathcal{M}(\mathbb{C}^{d_B}) \to \mathcal{M}(\mathbb{C}^{d_B})$ is CP, then for every state ρ_{AB} one has $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geq 0$.
- If $f : \mathcal{M}(\mathbb{C}^{d_B}) \to \mathcal{M}(\mathbb{C}^{d_B})$ is only positive, then for every separable state ρ_{AB}, one has $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geq 0$.
- Indeed,

$$[\mathrm{id}_{d_A} \otimes f] \left(\sum_i t_i \sigma_i^{(A)} \otimes \sigma_i^{(B)} \right) = \sum_i t_i \sigma_i^{(A)} \otimes f(\sigma_i^{(B)}) \geq 0,$$

since each term is positive semidefinite.

Entanglement detection via positive, but not CP maps

- Positive, but not CP maps f yield entanglement criteria: given ρ_{AB}, if $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \not\geq 0$, then ρ_{AB} is entangled.
- The following converse holds: if, for all positive, but not CP maps f, $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geq 0$, then ρ_{AB} is separable.

- The transposition map $\Theta(X) = X^\top$ is positive, but not CP. Let

$$\mathcal{PPT} := \{\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \,|\, [\mathrm{id}_{d_A} \otimes \Theta_{d_B}](\rho_{AB}) \geq 0\}.$$

- We have $\mathcal{SEP} \subseteq \mathcal{PPT}$, with equality iff

$$(d_A, d_B) \in \{(2,2), (2,3), (3,2)\}.$$

- This is the consequence of a deep result in operator algebra: every positive map $f : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_{2,3}(\mathbb{C})$ can be written as

$$f = g_1 + \Theta \circ g_2, \qquad \text{with } g_{1,2} \text{ CP}.$$

- Question: for large $d_{A,B}$ how much smaller is $\mathcal{SEP}$ than $\mathcal{PPT}$?

The PPT criterion at work

- Consider the Bell state $\rho_{AB} = |\psi\rangle\langle\psi|$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B).$$

- Written as a matrix in $\mathcal{M}_{2\cdot 2}^{1,+}(\mathbb{C})$

$$\rho_{AB} = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) = \frac{1}{2}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

- Partial transposition: transpose each block B_{ij}:

$$[\mathrm{id}_2 \otimes \Theta](\rho_{AB}) = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

- This matrix is no longer positive $\implies$ the state is entangled.

Quantum channels

Channels	Deterministic	Random mixture
Classical	$f : \{1, \ldots, d\} \to \{1, \ldots, d\}$	Q Markov (stochastic)
Quantum	$U \in \mathcal{U}(d)$	Φ CPTP map

- Quantum channels: CPTP maps $\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_{d'}(\mathbb{C})$
  - CP - complete positivity: $\Phi \otimes \mathrm{id}_r$ is a positive map, $\forall r \geq 1$
  - TP - trace preservation: $\mathrm{Tr} \circ \Phi = \mathrm{Tr}$.
- Example 1: unitary conjugation $\Phi(X) = UXU^*$ for a unitary matrix $U \in \mathcal{U}(d)$.
- Example 2: depolarizing channel $\Delta(X) = (\mathrm{Tr}\, X)\frac{I}{d}$.

Structure of CPTP maps

Theorem (Stinespring-Kraus-Choi)

Let $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ be a linear map. The following assertions are equivalent:

1. *The map Φ is completely positive and trace preserving.*
2. *There exist an integer n ($n = d^2$ suffices) and an isometry $V : \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^n$ such that*
$$\Phi(X) = [\mathrm{id}_d \otimes \mathrm{Tr}_n](VXV^*).$$
3. *There exist operators $A_1, \ldots, A_n \in \mathcal{M}_d(\mathbb{C})$ satisfying $\sum_i A_i^* A_i = I_d$ such that*
$$\Phi(X) = \sum_{i=1}^{n} A_i X A_i^*.$$
4. *The Choi matrix C_Φ is positive semidefinite, where*
$$C_\Phi := \sum_{i,j=1}^{d} E_{ij} \otimes \Phi(E_{ij}) \in \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C}).$$

(Minimum Output) Entropy

- von Neumann and Rényi entropies of quantum states

$$H(\rho) = H^1(\rho) = -\operatorname{Tr}(\rho \log \rho) \qquad H^p(\rho) = \frac{\log \operatorname{Tr} \rho^p}{1-p}, \quad p > 0.$$

- Entropies are additive

$$H^p(\rho_A \otimes \rho_B) = H^p(\rho_A) + H^p(\rho_B).$$

- p-Minimal Output Entropy of a quantum channel

$$H^p_{\min}(\Phi) = \min_{\rho \in \mathcal{M}_d^{1,+}(\mathbb{C})} H^p(\Phi(\rho)) = \min_{x \in \mathbb{C}^d, \, \|x\|=1} H^p(\Phi(|x\rangle\langle x|)).$$

- Is the p-MOE additive:

$$\forall \Phi, \Psi \qquad H^p_{\min}(\Phi \otimes \Psi) = H^p_{\min}(\Phi) + H^p_{\min}(\Psi) \quad ?$$

- NO!!!
  - $p > 1$: Hayden + Winter '08;
  - $p = 1$: Hastings '09.
- Why care? Simple formula for the classical capacity of quantum channels: if additivity holds, then there is no need to use inputs entangled over multiple uses of Φ.

États quantiques aléatoires et la transposée partielle

Probability measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- We want to measure volumes of subsets of $\mathcal{M}_d^{1,+}(\mathbb{C})$, with $d = d_A d_B$.
- A natural choice is to use the Lebesgue measure (see $\mathcal{M}_d^{1,+}(\mathbb{C})$ as a compact subset of $\mathcal{M}_d^{sa}(\mathbb{C})$). The set of separable states $\mathcal{SEP}$ has positive volume, since $\mathcal{SEP}$ contains an open ball around I/d.
- Another choice - open quantum systems point of view: assume your system Hilbert space $\mathbb{C}^d = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ is coupled to an environment $\mathbb{C}^{d_C}$.
- On the tri-partite system $\mathcal{H}_{ABC} = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_C}$, consider a random pure state $|\psi\rangle_{ABC}$, i.e. a uniform random point on the unit sphere of the total Hilbert space $\mathcal{H}_{ABC}$. Trace out the environment $\mathbb{C}^{d_C}$ to get a *random density matrix*

$$\rho_{AB} = [\mathrm{id}_A \otimes \mathrm{id}_B \otimes \mathrm{Tr}_C] |\psi\rangle\langle\psi|_{ABC}.$$

- These probability measures have been introduced by Życzkowski and Sommers and they are called the induced measures of parameters $d = d_A d_B$ and $s = d_C$; we denote them by $\mu_{d,s}$.
- Remarkably, the Lebesgue measure is obtained for $s = d$.

Probability measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- Here's an equivalent way of defining the measures $\mu_{d,s}$, in the spirit of Random Matrix Theory.
- Let $X \in \mathcal{M}_{d\times s}(\mathbb{C})$ be a $d \times s$ matrix with i.i.d. complex standard Gaussian entries (i.e. a Ginibre random matrix). Define

$$W_{d,s} = XX^* \text{ and } \mathcal{M}^{1,+}(\mathbb{C}^d) \ni \rho_{d,s} = \frac{XX^*}{\mathrm{Tr}(XX^*)} = \frac{W_{d,s}}{\mathrm{Tr}\, W_{d,s}}.$$

- The random matrix $W_{d,s}$ is called a Wishart matrix and the distribution of $\rho_{d,s}$ is precisely $\mu_{d,s}$.
- The measure $\mu_{d,s}$ is unitarily invariant: if $\rho \sim \mu_{d,s}$ and U is a fixed unitary matrix, then $U\rho U^* \sim \mu_{d,s}$.
- Density of $\mu_{d,s}$: $\mathrm{d}\mathbb{P}(\rho) = C_{d,s} \det(\rho)^{s-d} \mathbf{1}_{\rho \geq 0,\, \mathrm{Tr}\, \rho = 1}\, \mathrm{d}\rho$.
- Integrating out the eigenvectors, we obtain the eigenvalue density formula for random quantum states:

$$\mathrm{d}\mathbb{P}(\lambda_1, \ldots, \lambda_d) = C'_{d,s} \left[\prod_i \lambda_i^{s-d}\right] \left[\prod_{i<j} (\lambda_i - \lambda_j)^2\right] \mathbf{1}_{\lambda_i \geq 0,\, \sum_i \lambda_i = 1}\, \mathrm{d}\lambda.$$

Eigenvalues for induced measures

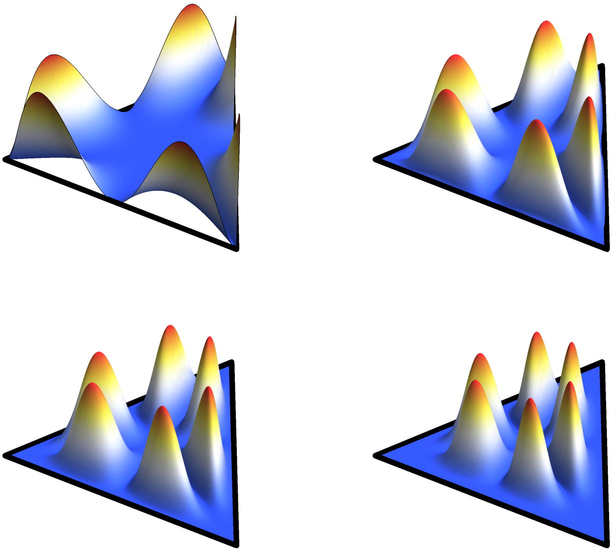


Figure: Induced measures for $d = 3$ and $s = 3, 5, 7, 10$.

Eigenvalues for induced measures

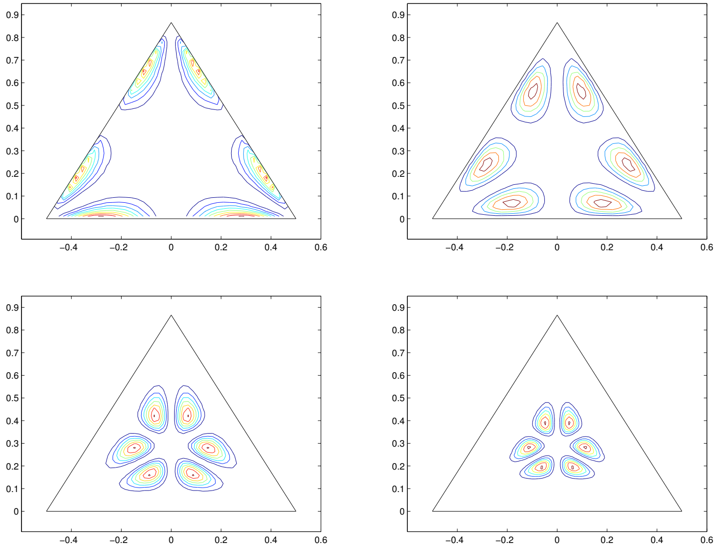


Figure: Induced measures for $d = 3$ and $s = 3, 5, 7, 10$.

Volume of convex sets under the induced measures

- Fix d, and let $C \subset \mathcal{M}^{1,+}(\mathbb{C}^d)$ a convex body, with $I_d/d \in \mathrm{int}(C)$. Then
$$\lim_{s\to\infty} \mu_{d,s}(C) = 1.$$
In other words, the eigenvalues of a random density matrix $\rho_{AB} \sim \mu_{d,s}$ with d fixed and $s \to \infty$ converge to I/d.

Definition

A pair of functions $(s_0(d), s_1(d))$ are called a threshold for a family of convex sets $(K_d)_d$ if both conditions below hold

- If $s(d) \lesssim s_0(d)$, then
$$\lim_{d\to\infty} \mu_{d,s(d)}(K_d) = 0;$$
- If $s(d) \gtrsim s_1(d)$, then
$$\lim_{d\to\infty} \mu_{d,s(d)}(K_d) = 1.$$

Thresholds for entanglement criteria

- Below, the threshold functions $s_{0,1}(d)$ are of the form

$$s_0(d) = s_1(d) = cd; \qquad \text{we put } r := \min(d_A, d_B).$$

Crit. \ Reg.	$d_A = d_B \to \infty$	$d_B \to \infty$	$d_A \to \infty$
$\mathcal{SEP}$	$\infty \ (r \lesssim c \lesssim r\log^2 r)$	?	?
$\mathcal{PPT}$	4	$2 + 2\sqrt{1 - \frac{1}{r^2}}$	$2 + 2\sqrt{1 - \frac{1}{r^2}}$

- The results in the table above can be interpreted in the following way: for a convex set K having a threshold c, a random density matrix $\rho_{AB} \sim \mu_{d,s}$ with large s, d will satisfy
  - If $s/d > c$, $\mathbb{P}[\rho_{AB} \in K] \approx 1$
  - If $s/d < c$, $\mathbb{P}[\rho_{AB} \in K] \approx 0$.
- In particular, in the regime $d_A = d_B \to \infty$, $s \sim cd$ with $c > 4$, random quantum states are PPT and entangled! In other words, in this regime, the PPT criterion is very weak.

Wishart matrices

Theorem (Marcenko-Pastur)

Let W be a complex Wishart matrix of parameters (d, cd). Then, almost surely with $d \to \infty$, the empirical spectral distribution of W/d converges in moments to a free Poisson distribution (a.k.a. Marčenko-Pastur distribution) π_c of parameter c.

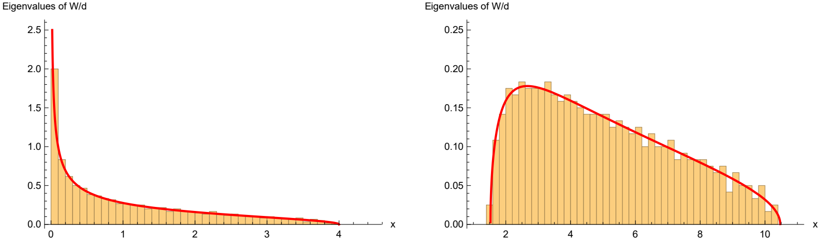


Figure: Eigenvalue distribution for Wishart matrices. In blue, the density of theoretical limiting distribution, π_c. In the two pictures, $d = 1000$, and $c = 1, 5$.

Partial transposition of a Wishart matrix

Theorem (Banica, N.)

Let W be a complex Wishart matrix of parameters (dn, cdn). Then, almost surely with $d \to \infty$, the empirical spectral distribution of $[\mathrm{id} \otimes \Theta](W_{AB}/d)$ converges in moments to a free difference of free Poisson distributions of respective parameters $cn(n \pm 1)/2$.

Corollary

The limiting measure above has positive support iff

$$c > c_{PPT} := 2 + 2\sqrt{1 - \frac{1}{n^2}}.$$

Partial transposition criterion - numerics

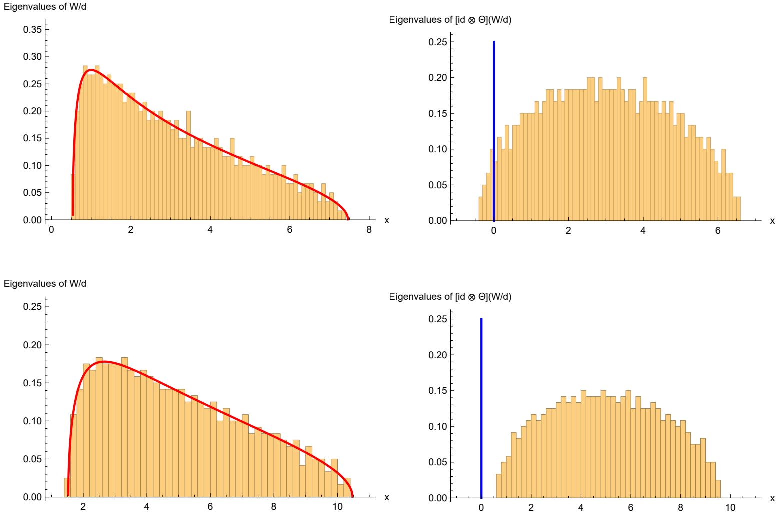


Figure: Wishart matrices before (left) and after (right) the application of the partial transposition. Here, $d = d_A = 200$, $n = d_B = 3$, and $c = 3$ (top), $c = 5$ (bottom). Note that $3 < c_{PPT}^{n=3} = 3.88562 < 5$.

Canaux quantiques aléatoires et la propriété d'additivité

Random quantum channels

- Counterexamples to additivity conjectures are random.
- Random quantum channels from random isometries

$$\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C}), \qquad \Phi(\rho) = [\mathrm{id}_k \otimes \mathrm{Tr}_n](V \rho V^*),$$

where V is a Haar random partial isometry

$$V : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n.$$

- We shall assume that $n \to \infty$, $d \sim tkn$, and k, t are fixed parameters.

- How to get counterexamples to the additivity conjecture?
- Choose Φ to be random and $\Psi = \bar{\Phi}$; this way, $H^p_{\min}(\Psi) = H^p_{\min}(\Phi)$.
- Bound

$$H^p_{\min}(\Phi \otimes \bar{\Phi}) \leq B_2 < 2B_1 \leq 2H^p_{\min}(\Phi).$$

- Bound B_2: choose the maximally entangled input and upper bound the entropy of the output.

Strategy for B_1

- Remember: we want
$$H^p_{\min}(\Phi \otimes \bar{\Phi}) \leq B_2 < 2B_1 \leq 2H^p_{\min}(\Phi).$$
- We shall do more: we compute the exact limit (as $n \to \infty$) of $H^p_{\min}(\Phi)$.

Theorem (Belinschi, Collins, N. '13)

For all $p \geq 1$, almost surely,
$$\lim_{n\to\infty} H_p^{min}(\Phi) = H_p(a, \underbrace{b, b, \ldots, b}_{k-1}),$$
where a, b do not depend on p, $b = (1-a)/(k-1)$ and $a = \varphi(1/k, t)$ with
$$\varphi(s,t) = \begin{cases} s + t - 2st + 2\sqrt{st(1-s)(1-t)} & \text{if } s+t < 1; \\ 1 & \text{if } s+t \geq 1. \end{cases}$$

Entanglement of a subspace

- For a vector $x = \sum_{i=1}^{k} \sqrt{\lambda_i(x)} e_i \otimes f_i$, define $H(x) = H(\lambda(x)) = -\sum_i \lambda_i(x) \log \lambda_i(x)$, the entropy of entanglement of the bipartite pure state x.
- Note that
  1. The state x is separable, $x = e \otimes f$, iff $H(x) = 0$.
  2. The state x is maximally entangled, $x = k^{-1/2} \sum_i e_i \otimes f_i$, iff $H(x) = \log k$.
- For a subspace $V \subseteq \mathbb{C}^k \otimes \mathbb{C}^n$, define

$$\boxed{H_p^{\min}(V) = \min_{y \in V, \, \|y\|=1} H_p(y),}$$

  the minimal entanglement of vectors in V.
- Recall that we are interested in random isometries $V : \mathbb{C}^{tnk} \to \mathbb{C}^k \otimes \mathbb{C}^n$ and the channels Φ they define. It turns out that $H_p^{\min}(\Phi) = H_p^{\min}(\text{Ran } V)$, so we focus on subspace entanglement from now on.

Singular values of vectors from a subspace

- Entropy is just a statistic, look at the set of all singular values directly!
- For a subspace $V \subset \mathbb{C}^k \otimes \mathbb{C}^n$ of dimension $\dim V = d$, define the set eigen-/singular values or Schmidt coefficients

$$K_V = \{\lambda(x) \,:\, x \in V, \|x\| = 1\}.$$

- Example: the anti-symmetric subspace
  - Let $k = n$ and put $V = \Lambda^2(\mathbb{C}^n)$.
  - $\dim V = n(n-1)/2$.
  - Example of a vector in V:

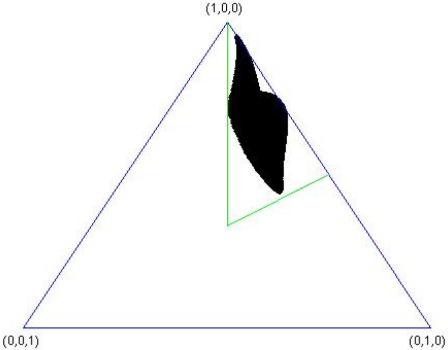
$$V \ni x = \frac{1}{\sqrt{2}}(e \otimes f - f \otimes e).$$

  - Fact: singular values of vectors in V come in pairs.
  - The least entropy vector in V is as above, with $e \perp f$ and $H(x) = \log 2$.
  - Thus, $H^{\min}(V) = \log 2$ and one can show that

$$K_V = \{(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots) \in \Delta_n \,:\, \lambda_i \geq 0, \sum_i \lambda_i = 1/2\}.$$

Examples - K_V

$V = \mathrm{span}\{G_1, G_2\}$, where $G_{1,2}$ are 3×3 independent Ginibre random matrices.



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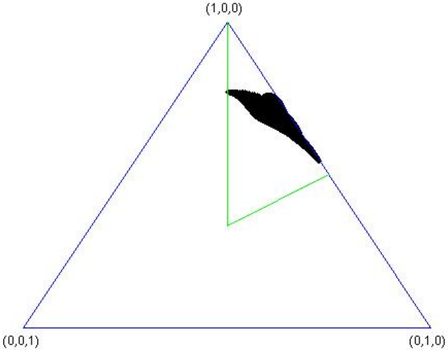

Examples - K_V

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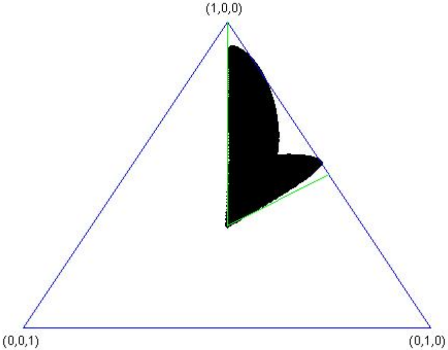

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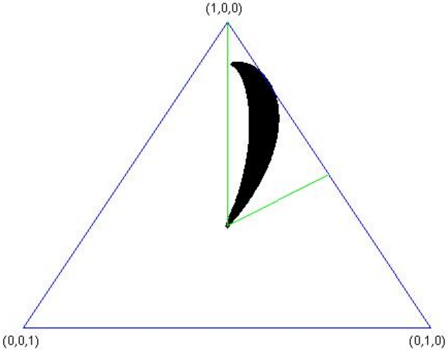

Random subspaces - the (t)-norm

- Recall that we are interested in random isometries/subspaces in the following asymptotic regime: k fixed, $n \to \infty$, and $d \sim tkn$, for a fixed parameter $t \in (0,1)$.

Theorem (Belinschi, Collins, N. '10)

For a sequence of uniformly distributed random subspaces V_n, the set K_{V_n} of singular values of unit vectors from V_n converges (almost surely, in the Hausdorff distance) to a deterministic, convex subset $K_{k,t}$ of the probability simplex Δ_k

$$K_{k,t} := \{\lambda \in \Delta_k \mid \forall x \in \Delta_k, \langle \lambda, x\rangle \leq \|x\|_{(t)}\}.$$

with the (t)-norm being defined as follows:

$$\|x\|_{(t)} := \|p_t x p_t\|_\infty,$$

where x an element having spectrum x, and p is a projection of trace t, free from x.

- If $t > 1 - 1/k$, $\|\cdot\|_{(t)} = \|\cdot\|_\infty$; also $\lim_{t\to 0^+} \|x\|_{(t)} = k^{-1}|\sum_i x_i|$.

An open problem

Find explicit (i.e. non-random) examples of subspaces $V \subset \mathbb{C}^k \otimes \mathbb{C}^n$ with

1. large $\dim V$;
2. large $H_{\min}(V)$.

Merci!

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