Random quantum states

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Open Quantum Systems meeting
Grenoble, November 29th, 2010

Random quantum states

— unstructured models —

Random pure quantum states

- Pure states of a finite dimensional quantum system: $|\psi\rangle \in \mathcal{H} \simeq \mathbb{C}^N$.
- Up to an unimportant phase, the set of pure states is the unit sphere of $\mathbb{C}^N$.
- For $N = 2$, a pure qubit is a point on the Bloch sphere (unit sphere of $\mathbb{R}^3$).
- A random pure state in $\mathcal{H} = \mathbb{C}^N$ is a uniform point on the unit sphere of $\mathbb{C}^N$.
- One can sample from this distribution by normalizing a vector of N i.i.d. complex Gaussian random variables, $|\psi\rangle = X/\|X\|_2$.
- Equivalent definition: let $U \in \mathcal{U}_N$ be a Haar-distributed random unitary matrix and let $|\varphi_0\rangle$ be a fixed quantum state. Then, $|\varphi\rangle = U|\varphi_0\rangle$ has the same distribution as $|\psi\rangle$.
- If, instead of a uniform distribution, we wand a random state "concentrated around" $|\varphi_0\rangle$, use $|\varphi_t\rangle = U_t|\varphi_0\rangle$, where U_t is a random unitary Brownian motion. In the limit $t \to \infty$, one recovers the previous model.
- The structure of $\mathcal{H}$ does not play any role here $\rightsquigarrow$ unstructured quantum states

Random pure states and the induced ensemble

- Induced ensemble : partial trace a random pure state on a composite system $\mathcal{H} \otimes \mathcal{K}$:
$$\rho = \mathrm{Tr}_K |\psi\rangle\langle\psi|,$$
where $|\psi\rangle$ is a random pure state on $\mathbb{C}^N \otimes \mathbb{C}^K$.
- The random matrix ρ has the same distribution as a rescaled Wishart matrix $W/\operatorname{Tr} W$, where $W = XX^*$ with X a Ginibre (i.i.d. Gaussian entries) matrix from $\mathcal{M}_{N\times K}(\mathbb{C})$.
- The eigenvalue density of ρ is given by
$$(\lambda_1, \ldots, \lambda_N) \mapsto C_{N,K} \prod_{i=1}^{N} \lambda_i^{K-N} \Delta(\lambda)^2,$$
where $\Delta(\lambda) = \prod_{1\leqslant i<j\leqslant N} (\lambda_i - \lambda_j)$.
- Exact formula for the average von Neumann entropy [Page, '95]
$$\mathbb{E}H(\rho) = \Psi(NK+1) - \Psi(K+1) - \frac{N-1}{2K} \sim \ln(N) - N/2K.$$

Random density matrices - asymptotics

- In the limit $N \to \infty$, $K \sim cN$, for a fixed constant $c > 0$, the empirical spectral distribution of the rescaled eigenvalues

$$L_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{cN\lambda_i},$$

converges almost surely to the Marchenko-Pastur distribution $\pi_c^{(1)}$.

- The Marchenko-Pastur (or free Poisson) distribution is defined by

$$\pi_c^{(1)} = \max\{1 - c, 0\}\delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \mathbf{1}_{[a,b]}(x)dx,$$

where $a = (\sqrt{c} - 1)^2$ and $b = (\sqrt{c} + 1)^2$.

Random density matrices - asymptotics

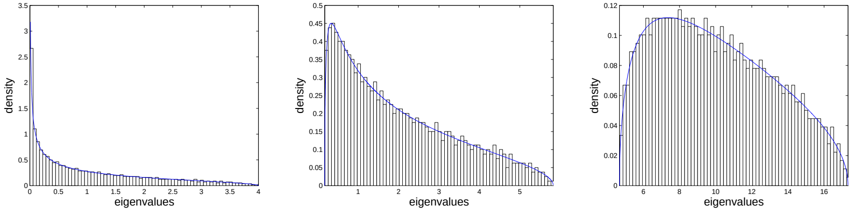


Figure: Empirical and limit measures for $(N = 1000, K = 1000)$, $(N = 1000, K = 2000)$ and $(N = 1000, K = 10000)$.

Random quantum states associated to graphs

— structured models —

Pure states associated to graphs

- Total Hilbert space has a product structure $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$.
- We want our randomness model to encode initial quantum correlation between different subsystems.
- The structure of correlations will be encoded in a graph:
  - Vertices encode the different subsystems;
  - Edges encode the presence of entanglement.

Pure states associated to graphs - examples

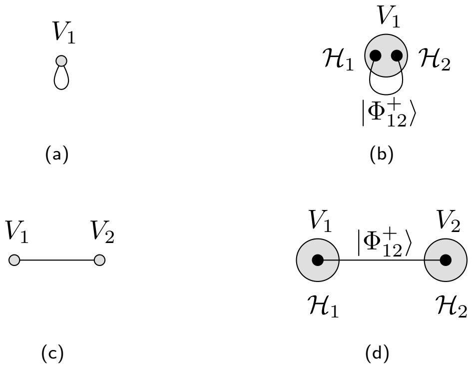


Figure: Graphs with one edge: a loop on one vertex, in simplified notation (a) and in the standard notation (b), and two vertices connected by one edge, in simplified notation (c) and in the standard notation (d).

Pure states associated to graphs - examples

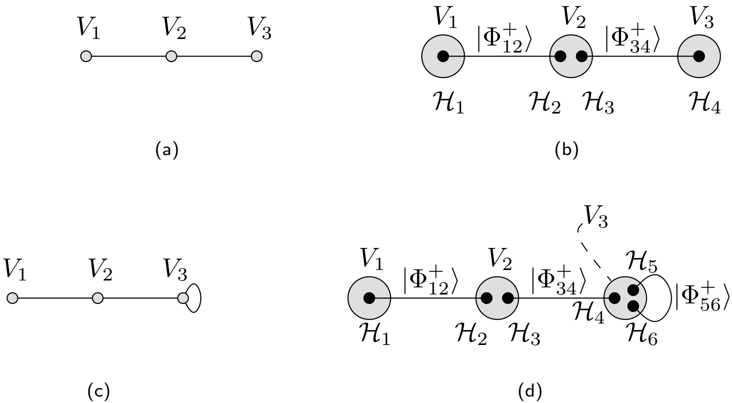


Figure: A linear 2-edge graph, in the simplified notation (a) and in the standard notation (b). Graph consisting of 3 vertices and 3 bonds (c), one of which is connected to the same vertex so it forms a loop; (d) the corresponding ensemble of random pure states defined in a Hilbert space composed of 6 subspaces represented by dark dots.

Pure states associated to graphs - formal definition

- Consider an undirected graph Γ consisting of m *edges* (or *bonds*) $B_1, \ldots, B_m$ and k *vertices* $V_1, \ldots V_k$.
- We associate to Γ a pure state $|\tilde{\Psi}\rangle\langle\tilde{\Psi}| \in \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{2m}$:

$$|\tilde{\Psi}\rangle = \bigotimes_{\{i,j\} \text{ edge}} |\Phi^+_{i,j}\rangle,$$

where $|\Phi^+_{i,j}\rangle$ denotes a maximally entangled state:

$$|\Phi^+_{ij}\rangle = \frac{1}{\sqrt{d_i N}} \sum_{x=1}^{d_i N} |e_x\rangle \otimes |f_x\rangle,$$

- $\dim \mathcal{H}_i = d_i N$, with d_i fixed parameters and $N \to \infty$. For each edge $\{i, j\}$, we have $d_i = d_j$.
- At each vertex, a Haar unitary matrix acts on the subsystems

$$\bigotimes_{i=1}^{n=2m} \mathcal{H}_i \ni |\Psi_\Gamma\rangle = \left(\bigotimes_{C \text{ vertex}} U_C \right) |\tilde{\Psi}\rangle$$

- The random unitary matrices $U_1, \ldots, U_k$ are independent.

Marginals of graph states

— moments and entropy —

Partial tracing random pure graph states

- Non-local properties of the random graph state $|\Psi\rangle \rightsquigarrow$ partition of the set of all $2m$ subsystems into two groups, $\{S, T\}$.
- Total Hilbert space can be decomposed as a tensor product, $\mathcal{H} = \mathcal{H}_T \otimes \mathcal{H}_S$.
- Reduced density operator

$$\rho_S = \mathrm{Tr}_T\, |\Psi\rangle\langle\Psi|.$$

- Graphically, partial traces are denoted at the graph by "crossing" the spaces $\mathcal{H}_i$ which are being traced out.

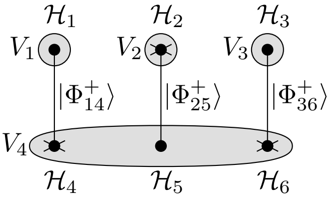


Figure: The random pure state supported on $n = 6$ subspaces is partial traced over the subspace $\mathcal{H}_T$ defined by the set $T = \{2, 4, 6\}$, represented by crosses.The reduced state ρ_S supported on subspaces corresponding to the set $S = \{1, 3, 5\}$.

Moments

- Use the method of moments: compute $\lim_{N\to\infty} \mathbb{E}\,\mathrm{Tr}(X^p)$ for a random matrix X.
- Using matrix coordinates, we can reduce our problem to computing integrals over the unitary group.

Theorem (Weingarten formula)

Let d be a positive integer and $\mathbf{i} = (i_1, \ldots, i_p)$, $\mathbf{i}' = (i'_1, \ldots, i'_p)$, $\mathbf{j} = (j_1, \ldots, j_p)$, $\mathbf{j}' = (j'_1, \ldots, j'_p)$ be p-tuples of positive integers from $\{1, 2, \ldots, d\}$. Then

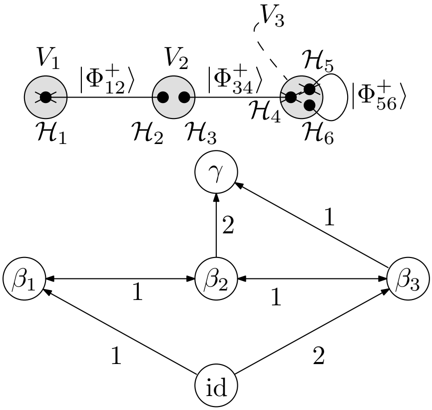
$$\int_{\mathcal{U}(d)} U_{i_1 j_1} \cdots U_{i_p j_p} \overline{U_{i'_1 j'_1}} \cdots \overline{U_{i'_p j'_p}}\, dU = \sum_{\alpha,\beta\in\mathcal{S}_p} \delta_{i_1 i'_{\alpha(1)}} \ldots \delta_{i_p i'_{\alpha(p)}} \delta_{j_1 j'_{\beta(1)}} \ldots \delta_{j_p j'_{\beta(p)}} \mathrm{Wg}(d, \alpha\beta^{-1}).$$

If $p \neq p'$ then

$$\int_{\mathcal{U}(d)} U_{i_1 j_1} \cdots U_{i_p j_p} \overline{U_{i'_1 j'_1}} \cdots \overline{U_{i'_{p'} j'_{p'}}}\, dU = 0.$$

Network associated to a marginal

- Using the Weingarten formula, one has to find the dominating term in a sum indexed by permutations of p objects.
- This optimization problem is equivalent to finding the maximum flow in a network.



Network associated to a marginal

- Network $(\mathcal{V}, \mathcal{E}, w)$ with vertex set $\mathcal{V}$, edge set $\mathcal{E}$ and edge capacities w.
- The vertex set $\mathcal{V} = \{\mathrm{id}, \gamma, \beta_1, \ldots, \beta_k\}$, with two distinguished vertices: the *source* id and the *sink* γ.
- The edges in $\mathcal{E}$ are oriented and they are of three types:

$$\mathcal{E} = \{(\mathrm{id}, \beta_i) \; ; \; |T_i| > 0\} \sqcup \{(\beta_i, \gamma) \; ; \; |S_i| > 0\} \sqcup \{(\beta_i, \beta_j), (\beta_j, \beta_i) \; ; \; |E_{ij}| > 0\},$$

  where S_i, T_i is are the surviving and traced out subsystems at vertex i and E_{ij} are the edges from vertex i to vertex j.
- The capacities of the edges are given by:

$$\begin{aligned} w(\mathrm{id}, \beta_i) &= |T_i| > 0 \\ w(\beta_i, \gamma) &= |S_i| > 0 \\ w(\beta_i, \beta_j) &= w(\beta_j, \beta_i) = |E_{ij}| > 0. \end{aligned}$$

Main result

Theorem (Collins, N., Życzkowski '10)

Asymptotically, as $N \to \infty$, the p-th moment of the reduced density matrix behaves as

$$\mathbb{E}\,\mathrm{Tr}(\rho_S^p) \sim N^{-X(p-1)} \cdot [\text{ combinatorial term } + o(1)],$$

where X is the maximum flow in the network associated to the marginal. The combinatorial part can be expressed in terms of the residual network obtained after removing the capacities of the edges that appear in the maximum flow solution.

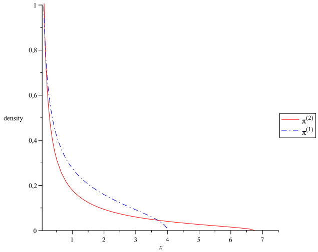
Fuss-Catalan limit distributions

Definition

- Matrix model: $\pi^{(s)}$ is the limit eigenvalue distribution of the random matrix $X_s = G_s \cdots G_2 G_1 G_1^* G_2^* \cdots G_s^*$, with i.i.d. Gaussian matrices G_i.
- Combinatorics: moments given by

$$\int x^p \, d\pi^{(s)}(x) = \frac{1}{sp+1}\binom{sp+p}{p}$$
$$= |\{\hat{0}_p \leqslant \sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_s \leqslant \hat{1}_p \in NC(p)\}|.$$

- Free probability: $\pi^{(s)} = \left(\pi^{(1)}\right)^{\boxtimes s}$, where $\pi^{(1)}$ is the free Poisson (or Marchenko-Pastur) distribution (of parameter $c = 1$).



Graph marginals with limit Fuss-Catalan distribution, $s = 1$

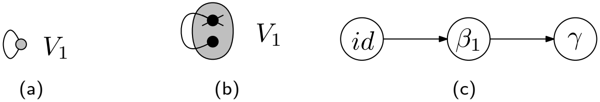
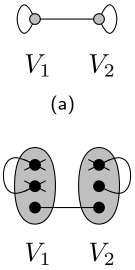


Figure: A vertex with one loop (a) and a marginal (b) having as a limit eigenvalue distribution the Marchenko-Pastur law $\pi^{(1)}$. In the network (c), both edges have capacity one.

- This is the simplest graph state having the Marchenko-Pastur asymptotic distribution.
- The reduced matrix is obtained by partial tracing an uniformly distributed pure state, hence it is an element of the *induced ensemble*.



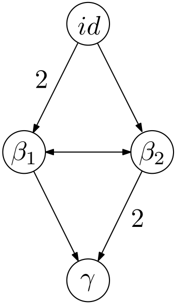


Figure: A graph (a) and a marginal (b) having as a limit eigenvalue distribution the Fuss-Catalan law $\pi^{(2)}$. In the network (c), non-labeled edges have capacity one. A maximum flow of 3 can be sent from the source id to the sink γ: one unit through each path id $\to \beta_i \to \gamma$, $i = 1, 2$ and one unit through the path id $\to \beta_1 \to \beta_2 \to \gamma$. In this way, the residual network is empty and the only constraint on the geodesic permutations β_1, β_2 is $\hat{0}_p \leqslant [\beta_1] \leqslant [\beta_2] \leqslant \hat{1}_p$, i.e. $[\beta_1]$ and $[\beta_2]$ form a 2-chain in $NC(p)$

Graph marginals with limit Fuss-Catalan distribution, $s \geqslant 2$

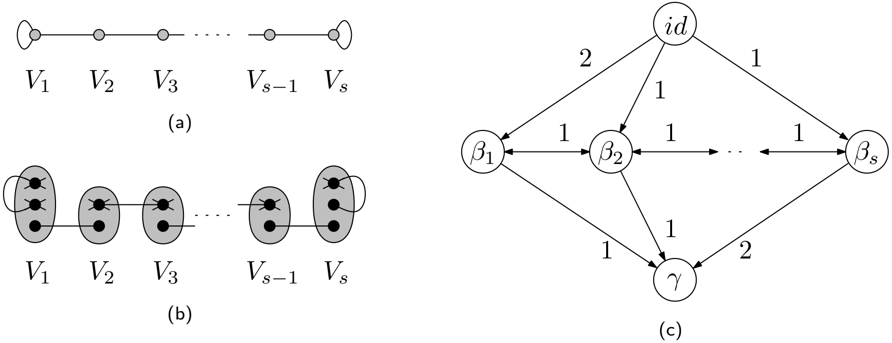


Figure: An example of a graph state (a) with a marginal (b) having as a limit eigenvalue distribution the s-th Fuss-Catalan probability measure $\pi^{(s)}$. The associated network (c) has a maximal flow of $s+1$, obtained by sending a unit of flow through each β_i and a unit through the path id $\to \beta_1 \to \cdots \to \beta_s \to \gamma$. The linear chain condition $[\beta_1] \leqslant \cdots \leqslant [\beta_s]$ follows.

Area laws

Area law holds for adapted marginals of graph states

- Setting: quantum many-body problem with *local interactions*
- "Area law" : the entanglement entropy of ground states grows like the boundary area of the subregion
- Non-extensive behavior for the entanglement entropy.
- A marginal ρ_S is called *adapted* if the number of traced out systems in each vertex is either zero or maximal.

Theorem

Let ρ_S be an adapted marginal of a graph state $|\Psi\rangle\langle\Psi|$. Then

$$H(\rho_S) = |\partial S| \log N$$

for all N. The boundary ∂S contains all the edges between the "traced out" vertices and the "surviving" vertices.

Area law holds for adapted marginals of graph states

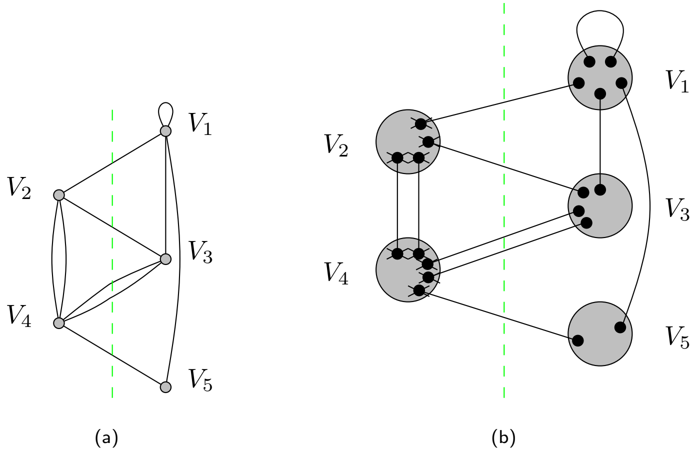


Figure: An example of a graph state (a) with an adapted marginal (b). The green dashed line represents the *boundary* between the traced and the surviving subsystems.

Area law holds for adapted marginals of graph states

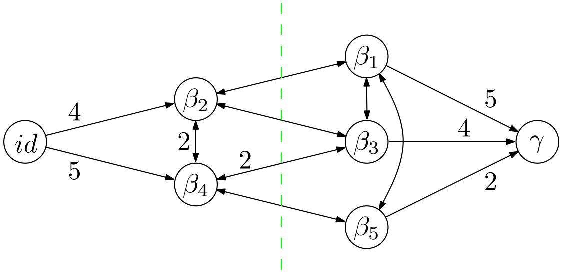


Figure: The network associated to an adapted marginal. Nodes cannot be connected to both the source and the sink. The maximum flow equals the minimum cut in the network which is the number of edges in the boundary ∂S.

Proof techniques

— graphical Weingarten calculus —

Method of moments & unitary integration

- Recall the main Theorem

Theorem

Asymptotically, as $N \to \infty$, the p-th moment of the reduced density matrix behaves as

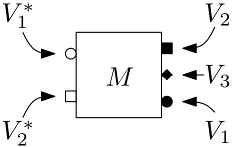
$$\mathbb{E}\,\mathrm{Tr}(\rho_S^p) \sim N^{-X(p-1)} \cdot [\ \textit{combinatorial term}\ + o(1)],$$

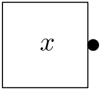
where X is the maximum flow in the network associated to the marginal. The combinatorial part can be expressed in terms of the residual network obtained after removing the capacities of the edges that appear in the maximum flow solution.

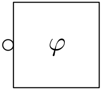
- Use the method of moments: compute $\lim_{N\to\infty} \mathbb{E}\,\mathrm{Tr}(\rho_S^p)$ for a random graph state ρ_S.
- Using matrix coordinates, we can reduce our problem to computing integrals over the unitary group.

Boxes & wires

- Graphical formalism inspired by works of Penrose, Coecke, Jones, etc.
- Tensors $\rightsquigarrow$ decorated boxes.

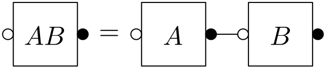


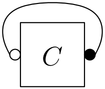




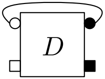
$M \in V_1 \otimes V_2 \otimes V_3 \otimes V_1^* \otimes V_2^*$ $\qquad$ $x \in V_1$ $\qquad$ $\varphi \in V_1^*$

- Tensor contractions (or traces) $V \otimes V^* \to \mathbb{C} \rightsquigarrow$ wires.



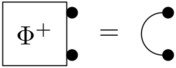


$\mathrm{Tr}(\mathrm{C})$



$\mathrm{Tr}_{\mathrm{V}_1}(\mathrm{D})$

- Bell state $\Phi^+ = \sum_{i=1}^{\dim V_1} e_i \otimes e_i \in V_1 \otimes V_1$



Graphical representation of random graph states

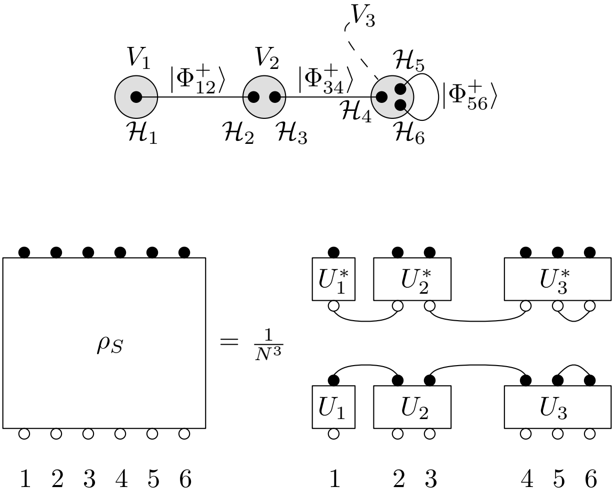


Figure: A graph state and its graphical representation.

Graphical representation of random graph states

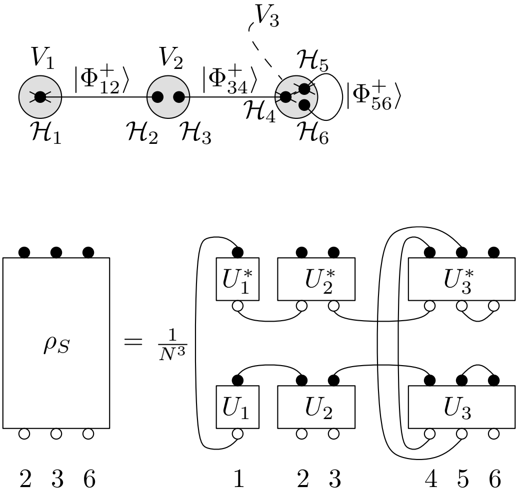


Figure: A marginal ρ_S of a graph state and its graphical representation.

Recall the Weingarten formula

Theorem (Weingarten formula)

Let d be a positive integer and $\mathbf{i} = (i_1, \ldots, i_p)$, $\mathbf{i}' = (i'_1, \ldots, i'_p)$, $\mathbf{j} = (j_1, \ldots, j_p)$, $\mathbf{j}' = (j'_1, \ldots, j'_p)$ be p-tuples of positive integers from $\{1, 2, \ldots, d\}$. Then

$$\int_{\mathcal{U}(d)} U_{i_1 j_1} \cdots U_{i_p j_p} \overline{U_{i'_1 j'_1}} \cdots \overline{U_{i'_p j'_p}} \, dU = \sum_{\alpha, \beta \in \mathcal{S}_p} \delta_{i_1 i'_{\alpha(1)}} \ldots \delta_{i_p i'_{\alpha(p)}} \delta_{j_1 j'_{\beta(1)}} \ldots \delta_{j_p j'_{\beta(p)}} \mathrm{Wg}(d, \alpha\beta^{-1}).$$

If $p \neq p'$ then

$$\int_{\mathcal{U}(d)} U_{i_1 j_1} \cdots U_{i_p j_p} \overline{U_{i'_1 j'_1}} \cdots \overline{U_{i'_{p'} j'_{p'}}} \, dU = 0.$$

- There is a graphical way of reading this formula on the diagrams !

"Graphical" Weingarten formula: graph expansion

Consider a diagram $\mathcal{D}$ containing random unitary matrices/boxes U and U^*.
Apply the following removal procedure:

1. Start by replacing U^* boxed by $\overline{U}$ boxes (by reversing decoration shading).
2. By the (algebraic) Weingarten formula, if the number p of U boxes is different from the number of $\overline{U}$ boxes, then $\mathbb{E}\mathcal{D} = 0$.
3. Otherwise, choose a pair of permutations $(\alpha, \beta) \in \mathcal{S}_p^2$. These permutations will be used to pair decorations of $U/\overline{U}$ boxes.
4. For all $i = 1, \ldots, p$, add a wire between each white decoration of the i-th U box and the corresponding white decoration of the $\alpha(i)$-th $\overline{U}$ box. In a similar manner, use β to pair black decorations.
5. Erase all U and $\overline{U}$ boxes. The resulting diagram is denoted by $\mathcal{D}_{(\alpha,\beta)}$.

Theorem (Collins, N. - CMP '10)

$$\mathbb{E}\mathcal{D} = \sum_{\alpha,\beta} \mathcal{D}_{(\alpha,\beta)} \,\mathrm{Wg}(d, \alpha\beta^{-1}).$$

First example

- Compute $\mathbb{E}|u_{ij}|^2 = \int_{\mathcal{U}(N)} |u_{ij}|^2 \, \mathrm{d}U$.

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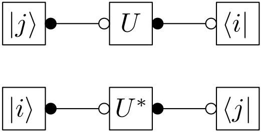

Figure: Diagram for $|u_{ij}|^2 = U_{ij} \cdot (U^*)_{ji}$.

First example

- Compute $\mathbb{E}|u_{ij}|^2 = \int_{\mathcal{U}(N)} |u_{ij}|^2 \, \mathrm{d}U$.

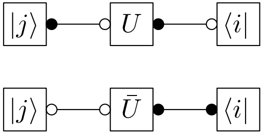

Figure: The U^* box replaced by an $\bar{U}$ box.

First example

- Compute $\mathbb{E}|u_{ij}|^2 = \int_{\mathcal{U}(N)} |u_{ij}|^2 \, \mathrm{d}U$.

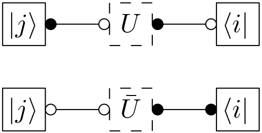

Figure: Erase U and $\bar{U}$ boxes.

First example

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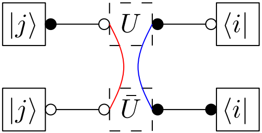


Figure: Pair white decorations (red wires) and black decorations (blue wires); only one possible pairing : $\alpha = (1)$ and $\beta = (1)$.

First example

- Compute $\mathbb{E}|u_{ij}|^2 = \int_{\mathcal{U}(N)} |u_{ij}|^2 \,\mathrm{d}U$.

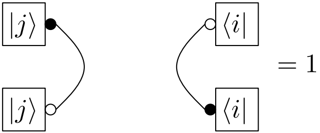
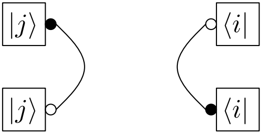


Figure: The only diagram $\mathcal{D}_{\alpha=(1),\beta=(1)} = 1$.

First example

- Compute $\mathbb{E}|u_{ij}|^2 = \int_{\mathcal{U}(N)} |u_{ij}|^2 \, \mathrm{d}U$.



$= 1$

Figure: The only diagram $\mathcal{D}_{\alpha=(1),\beta=(1)} = 1$.

- Conclusion :
  $\mathbb{E}|u_{ij}|^2 = \int |u_{ij}|^2 \, \mathrm{d}U = \mathcal{D}_{\alpha=(1),\beta=(1)} \cdot \mathrm{Wg}(N,(1)) = 1 \cdot 1/N = 1/N$.

Second example

- Compute $\mathbb{E}|u_{ij}|^4 = \int_{\mathcal{U}(N)} |u_{ij}|^4 \,\mathrm{d}U$.

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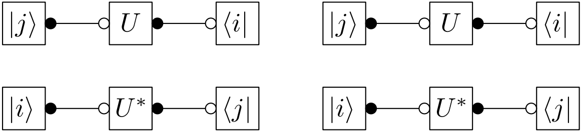

Figure: Diagram for $|u_{ij}|^2 = U_{ij} \cdot (U^*)_{ji}$.

Second example

- Compute $\mathbb{E}|u_{ij}|^4 = \int_{\mathcal{U}(N)} |u_{ij}|^4 \,\mathrm{d}U$.

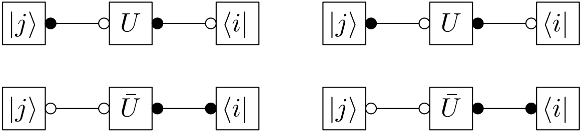

Figure: The U^* box replaced by an $\bar{U}$ box.

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- Compute $\mathbb{E}|u_{ij}|^4 = \int_{\mathcal{U}(N)} |u_{ij}|^4 \, \mathrm{d}U$.

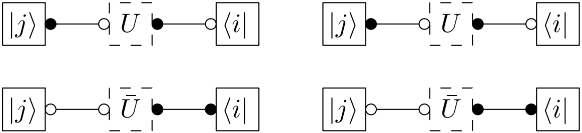

Figure: Erase U and $\bar{U}$ boxes.

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- Compute $\mathbb{E}|u_{ij}|^4 = \int_{\mathcal{U}(N)} |u_{ij}|^4 \, \mathrm{d}U$.

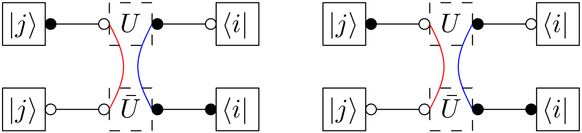

Figure: Pair white decorations (red wires) and black decorations (blue wires); first pairing : $\alpha = (1)(2)$ and $\beta = (1)(2)$.

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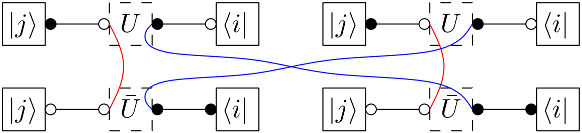

Figure: Second pairing : $\alpha = (1)(2)$ and $\beta = (12)$.

Second example

- Compute $\mathbb{E}|u_{ij}|^4 = \int_{\mathcal{U}(N)} |u_{ij}|^4 \, \mathrm{d}U$.

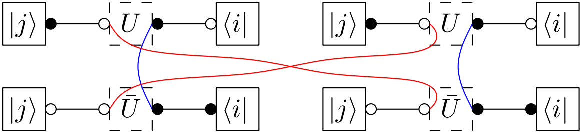

Figure: Third pairing : $\alpha = (12)$ and $\beta = (1)(2)$.

Second example

- Compute $\mathbb{E}|u_{ij}|^4 = \int_{\mathcal{U}(N)} |u_{ij}|^4 \,\mathrm{d}U$.

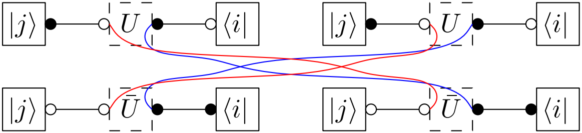

Figure: Fourth pairing : $\alpha = (12)$ and $\beta = (12)$.

Second example

- Compute $\mathbb{E}|u_{ij}|^4 = \int_{\mathcal{U}(N)} |u_{ij}|^4 \,\mathrm{d}U$.
- Conclusion :

$$\mathbb{E}|u_{ij}|^4 = \int |u_{ij}|^4 \,\mathrm{d}U =$$

$$\begin{aligned}
&\mathcal{D}_{(1)(2),(1)(2)} \cdot \mathrm{Wg}(N,(1)(2)) + \\
&\mathcal{D}_{(1)(2),(12)} \cdot \mathrm{Wg}(N,(12)) + \\
&\mathcal{D}_{(12),(1)(2)} \cdot \mathrm{Wg}(N,(12)) + \\
&\mathcal{D}_{(12),(12)} \cdot \mathrm{Wg}(N,(1)(2)) \\
&= \mathrm{Wg}(N,(1)(2)) + \mathrm{Wg}(N,(12)) + \mathrm{Wg}(N,(12)) + \mathrm{Wg}(N,(1)(2)) \\
&= \frac{2}{N^2-1} - \frac{2}{N(N^2-1)} = \frac{2}{N(N+1)}.
\end{aligned}$$

Third example : twirling

- Consider a fixed matrix $A \in \mathcal{M}_N(\mathbb{C})$. Compute $\int_{\mathcal{U}(N)} U^* A U \,\mathrm{d}U$.

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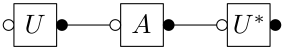

Figure: Diagram for $U^* A U$.

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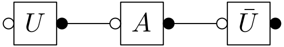

Figure: The U^* box replaced by an $\bar{U}$ box.

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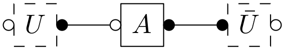

Figure: Erase U and $\bar{U}$ boxes.

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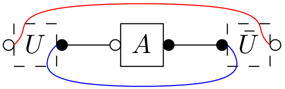

Figure: Pair white decorations (red wires) and black decorations (blue wires); only one possible pairing : $\alpha = (1)$ and $\beta = (1)$.

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Figure: The only diagram $\mathcal{D}_{\alpha=(1),\beta=(1)} = \mathrm{Tr}(A)\, \mathrm{I}_N$.

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- Consider a fixed matrix $A \in \mathcal{M}_N(\mathbb{C})$. Compute $\int_{\mathcal{U}(N)} U^* A U \, \mathrm{d}U$.

Figure: The only diagram $\mathcal{D}_{\alpha=(1),\beta=(1)} = \mathrm{Tr}(A)\, \mathrm{I}_N$.

- Conclusion : $\int_{\mathcal{U}(N)} U^* A U \, \mathrm{d}U = \mathcal{D}_{\alpha=(1),\beta=(1)} \cdot \mathrm{Wg}(N,(1)) = \frac{\mathrm{Tr}(A)}{N} \mathrm{I}_N$.

Conclusion and perspectives

Conclusion and perspectives

- Unstructured models
  1. Random pure states from unitary Brownian motion lead naturally to (non unitarily invariant) induced measures.
  2. Try to provide "environmental" models for measures on states defined geometrically, via distances (following Osipov and Życzkowski).

- Structured models
  1. Study lattice graphs.
  2. Connection with free probability: classical and free multiplicative convolution semigroups.
  3. More general area laws (in progress).
  4. Dual graphs: vertices are GHZ states and edges represent unitary coupling.

Thank you !

1. *Asymptotics of random density matrices* - Ann. Henri Poincaré 8 (2007), no. 8, 1521-1538.
2. *Random graph states, maximal flow and Fuss-Catalan distributions* (with Benoît Collins and Karol Zyczkowski) - J. Phys. A: Math. Theor. 43 (2010), 275303.
3. *Ensembles of structured random states and their entanglement* (with Benoît Collins, Karol Penson and Karol Zyczkowski) - arXiv preprint.
4. *Random quantum channels I: graphical calculus and the Bell state phenomenon* (with Benoît Collins) - Comm. Math. Phys. 297 (2010), no. 2, 345-370.
5. *Random pure quantum states via unitary Brownian motion* (with Clément Pellegrini) - in preparation.
6. *Area law for graph states* (with Benoît Collins and Karol Zyczkowski) - in preparation.