Graphical calculus for Random Quantum Channels

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Random quantum channels
&
additivity problems

Additivity for MOE of quantum channels

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- NO !!!
  - $p > 1$: Hayden '07, Hayden & Winter '08, Aubrun, Szarek & Werner '09
  - $p = 1$: Hastings '08, Fukuda & King '09, Horodecki & Brandao '09, Aubrun, Szarek & Werner '10

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- Holevo-Werner channel violates additivity of the p-Rényi entropy for $p > 4.79$. No known deterministic examples for $p = 1$ or p close to 1.
- Difficult, mathematically challenging problem.

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$$\Phi(\rho) = \mathrm{Tr}_{\mathrm{anc}}(V \rho V^*),$$

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- Equivalently, via the Stinespring dilation theorem

$$\Phi(\rho) = \mathrm{Tr}_{\mathrm{anc}}(U(\rho \otimes P_y)U^*),$$

where $y \in \mathbb{C}^{\frac{\mathrm{out} \times \mathrm{anc}}{\mathrm{in}}}$ and $U \in \mathcal{M}_{\mathrm{out} \times \mathrm{anc}}(\mathbb{C})$ is a Haar unitary matrix.

Models of interest

Finite rank output

- $\text{in} = tnk$,
- $\text{out} = k$,
- $\text{anc} = n$,

where $n, k \in \mathbb{N}$ and $t \in (0, 1)$. In general, we shall assume that

- $n \to \infty$ and k is fixed, but "large";
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Unbounded rank output

- in $= n$,
- out $= n$,
- anc $= k$,

where $n, k \in \mathbb{N}$ such that

- $n, k \to \infty$;
- $k/n \to c$, where $c > 0$ is a constant parameter.

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Strategy

- Use trivial bound

$$H^p_{\min}(\Phi \otimes \overline{\Phi}) \leqslant H^p\left([\Phi \otimes \overline{\Phi}](X_{12})\right),$$

for a particular choice of $X_{12} \in \mathcal{M}_{tnk}(\mathbb{C}) \otimes \mathcal{M}_{tnk}(\mathbb{C})$.

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- Bound entropies of the (random) density matrix

$$Z = [\Phi \otimes \overline{\Phi}](E_{\text{in}}) \in \mathcal{M}_{\text{out}}(\mathbb{C}) \otimes \mathcal{M}_{\text{out}}(\mathbb{C}).$$

Main result - finite rank output

Theorem (Collins + N. '09)

For all k, t, almost surely as $n \to \infty$, the eigenvalues of $Z = [\Phi \otimes \overline{\Phi}](E_{tnk})$ converge to

$$\left(t + \frac{1-t}{k^2}, \underbrace{\frac{1-t}{k^2}, \ldots, \frac{1-t}{k^2}}_{k^2-1 \text{ times}} \right).$$

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- Two improvements:
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- Precise knowledge of eigenvalues $\rightsquigarrow$ optimal estimates for entropies.
- However, smaller eigenvalues are the "worst possible".

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For all $c > 0$, the eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_{n^2}$ of $Z = [\Phi \otimes \overline{\Phi}](E_n)$ satisfy:

- *In probability, $cn\lambda_1 \to 1$.*
- *Almost surely, $\frac{1}{n^2-1} \sum_{i=2}^{n^2} \delta_{c^2 n^2 \lambda_i}$ converges to a free Poisson distribution of parameter c^2.*

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Free Poisson distribution

- The free Poisson distribution of parameter $c > 0$ is given by

$$\pi_c = \max(1-c, 0)\delta_0 + \frac{\sqrt{4c - (x-1-c)^2}}{2\pi x} \mathbf{1}_{[1+c-2\sqrt{c}, 1+c+2\sqrt{c}]}(x)\, dx.$$

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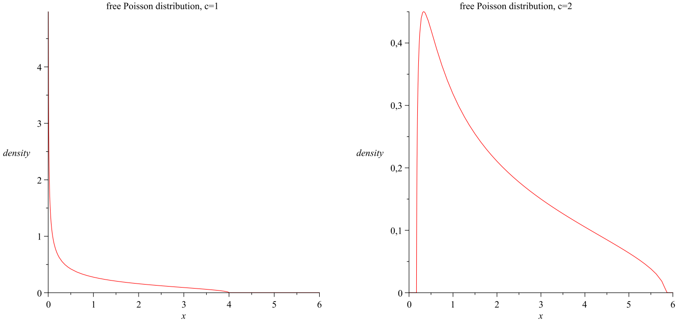

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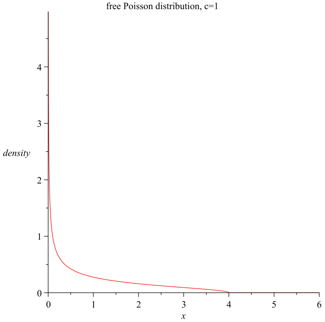

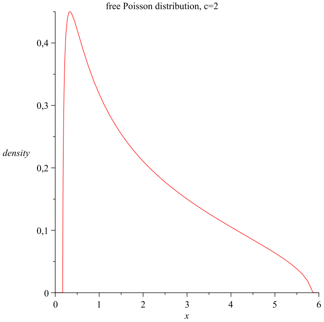

- Free Poisson Central Limit Theorem:

$$\left[\left(1-\frac{c}{n}\right)\delta_0 + \frac{c}{n}\delta_1\right]^{\boxplus n} \to \pi_c.$$

Independent channels - unbounded rank

Theorem (Collins + N. '09)

For $c > 0$, consider two independent *random quantum channels Φ and Ψ. The eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_{n^2}$ of $Z = [\Phi \otimes \Psi](E_n)$ are such that almost surely,*

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  - $\rightsquigarrow$ less uniform spectrum, lower entropy, better lower bounds: $H_{\min}^{\lambda_1} \leqslant 2\log n - \log n/n + o(\log n/n)$ vs. $H_{\min} \leqslant 2\log n - 1/2c^2 + o(1)$.
  - The first order of the entropy defect is given by the $n^2 - 1$ small eigenvalues, and not by the largest eigenvalue.

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  - Counter-examples for p-Rényi entropy additivity $\forall\ p > 1$ for $t = 1/2$ (input is coupled to a qubit).

- Unbounded rank outputs
  - "Large eigenvalue" bound : $\lambda_1 = 1/k \approx 1/(cn)$, $\lambda_2 = \ldots = \lambda_{k^2} = (1-1/k)/(n^2-1)$ spectrum;
  - Full, exact, asymptotic spectrum: $\lambda_1 = 1/(cn)$, $\lambda_2, \ldots, \lambda_{k^2}$ of order n^{-2}, distributed as π_{c^2};
  - $\rightsquigarrow$ less uniform spectrum, lower entropy, better lower bounds: $H_{\min}^{\lambda_1} \leqslant 2\log n - \log n/n + o(\log n/n)$ vs. $H_{\min} \leqslant 2\log n - 1/2c^2 + o(1)$.
  - The first order of the entropy defect is given by the $n^2 - 1$ small eigenvalues, and not by the largest eigenvalue.
  - No need for the conjugate channel trick, one may use independent channels !!!

Graphical calculus for random quantum channels

Boxes & wires

- Graphical formalism inspired by works of Penrose, Coecke, Jones, etc.

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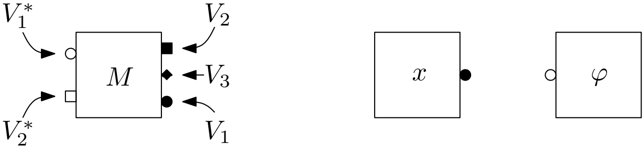

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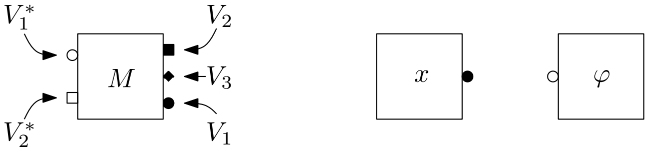

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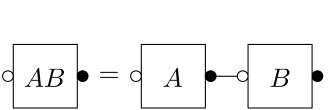

$\mathrm{Tr(C)}$ $\qquad \mathrm{Tr}_{\mathrm{V}_1}\mathrm{(D)}$

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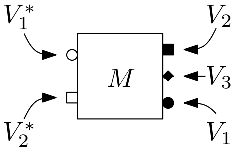

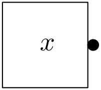

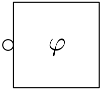

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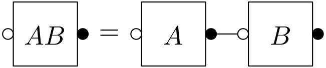
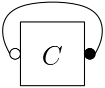
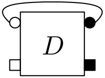


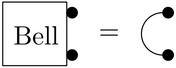


$\mathrm{Tr(C)}$



$\mathrm{Tr_{V_1}(D)}$

- Bell state $\mathrm{Bell} = \sum_{i=1}^{\dim V_1} e_i \otimes e_i \in V_1 \otimes V_1$

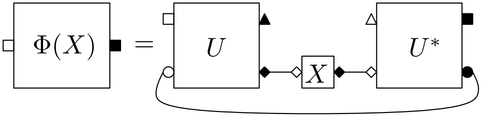


Graphical representation of quantum channels

- Decorations/labels

$$\begin{matrix}\bullet\\ \circ\end{matrix} = \mathbf{C}^n \qquad \begin{matrix}\blacksquare\\ \square\end{matrix} = \mathbf{C}^k \qquad \begin{matrix}\blacklozenge\\ \lozenge\end{matrix} = \mathbf{C}^{tnk} \qquad \begin{matrix}\blacktriangle\\ \triangle\end{matrix} = \mathbf{C}^{t^{-1}}$$

- Single channel (finite rank output)



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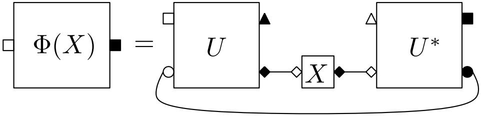
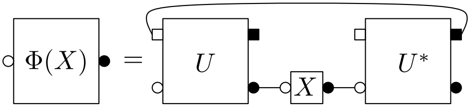


- Single channel (unbounded rank output, $n, k \to \infty$)



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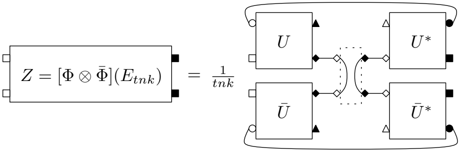

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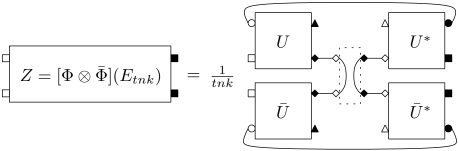

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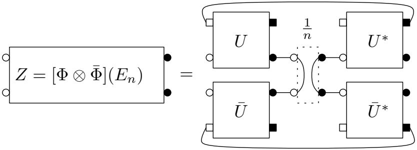

The Hayden-Winter trick: a graphical perspective

The Hayden-Winter trick

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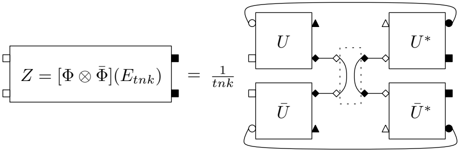

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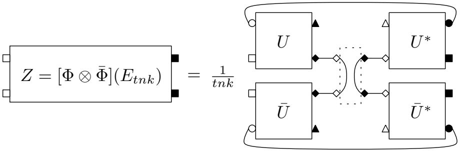

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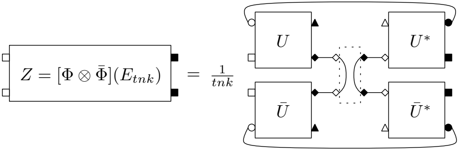

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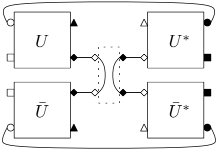
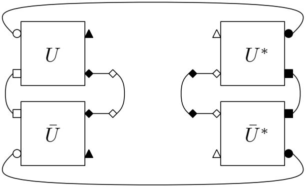
$$Z = [\Phi \otimes \bar{\Phi}](E_{tnk}) \quad = \quad \frac{1}{tnk}$$



- Want to show that Z has a large ($\geqslant t$) eigenvalue. 1st idea: find unit vector x such that $\langle x, Zx \rangle$ is big. Take $x = \mathrm{Bell}_{k^2}$.

$$\lambda_1(Z) \geq \langle \mathrm{Bell}_{k^2}, Z\mathrm{Bell}_{k^2} \rangle \; = \; \frac{1}{k}\frac{1}{tnk}$$



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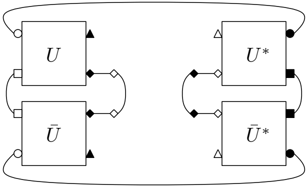

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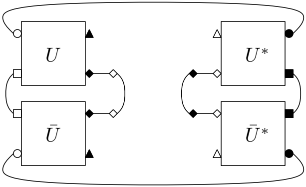
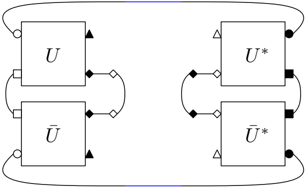
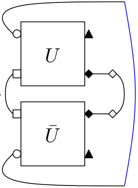
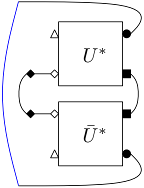
$$\lambda_1(Z) \geq \langle \mathrm{Bell}_{k^2}, Z\mathrm{Bell}_{k^2} \rangle = \frac{1}{k}\frac{1}{tnk}$$ 

- 2nd idea: $\mathrm{I}_n \geqslant E_n$

$\frac{1}{tnk^2}$  $\geq \frac{1}{tnk^2}$  $\frac{1}{n}$  $= \frac{1}{tn^2k^2} a^2$

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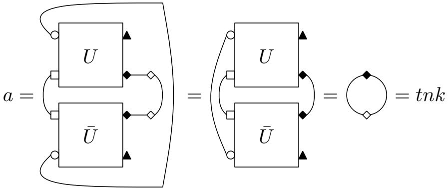

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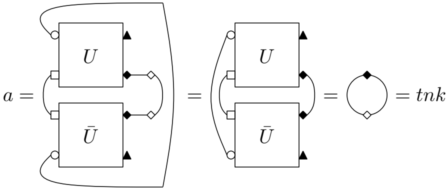

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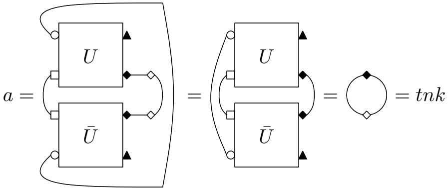

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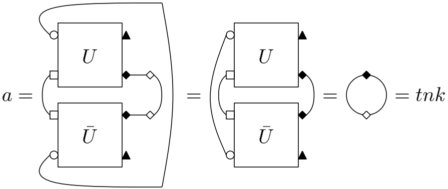

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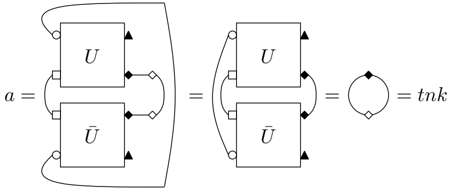

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- Is the choice of the Bell state as an input optimal ? Perhaps not...
- Possible improvement: choose an input adapted to the channel: $X_{12} = f(U)$ (*work in progress with Benoit and Motohisa*).

Dealing with random boxes: graphical Weingarten formula

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- Example (finite rank)

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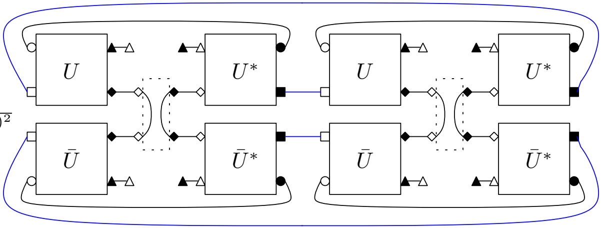

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- There is a graphical way of reading this formula on the diagrams !

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5. Erase all U and $\overline{U}$ boxes. The resulting diagram is denoted by $\mathcal{D}_{(\alpha,\beta)}$.

Theorem

$$\mathbb{E}\mathcal{D} = \sum_{\alpha,\beta} \mathcal{D}_{(\alpha,\beta)} \operatorname{Wg}(d, \alpha\beta^{-1}).$$

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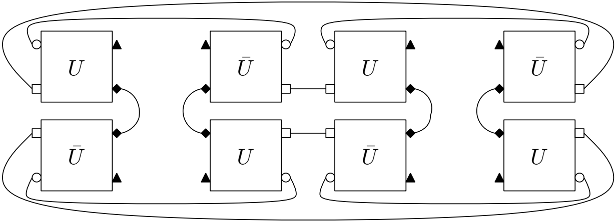
The original diagram



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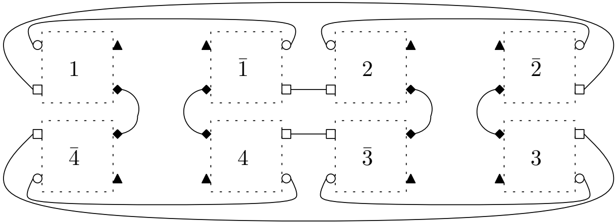
The diagram with the boxes removed



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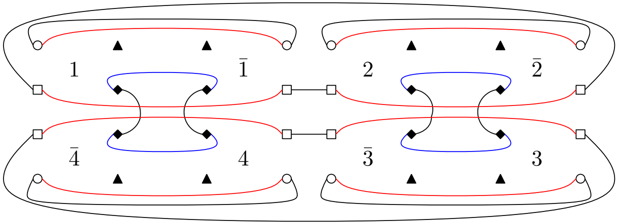

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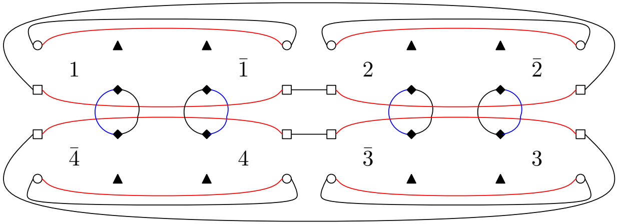

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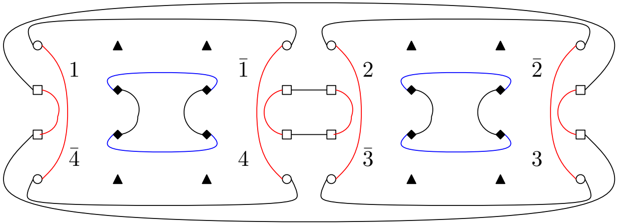

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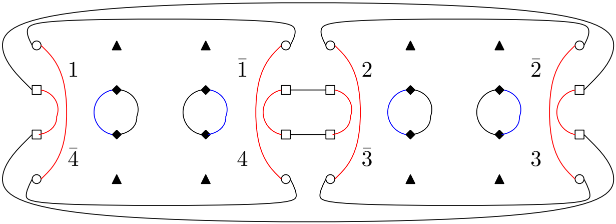

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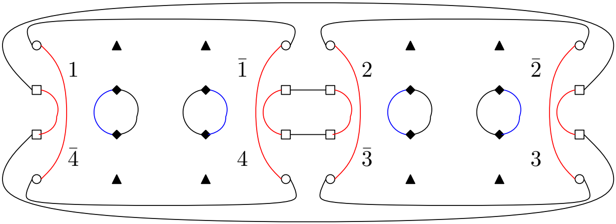

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- Contributions of diagrams $\rightsquigarrow$ counting the loops $\rightsquigarrow$ statistics over permutations.

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- The unbounded rank case for conjugate channels is more delicate, since the $n^2 - 1$ smaller eigenvalues are one order of magnitude below the largest eigenvalue. When computing moments of the matrix Z, only the large ($\sim n^{-1}$) eigenvalue gives a contribution. One needs to consider the eigenspace compression QZQ, where $Q = \mathrm{I} - E_n$ and finally apply interlacing results for eigenvalues.

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- Geodesic problems in symmetric groups $\Rightarrow$ non-crossing partitions $\Rightarrow$ free probability.
- The free Poisson distribution is characterized by its moments:

$$\int x^p \, d\pi_c(x) = \sum_{\substack{\alpha \in \mathcal{S}_p \\ \#\alpha + \#(\gamma^{-1}\alpha) = p+1}} c^{\#\alpha}.$$

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- Other applications to QIT (with K. Życzkowski: structured random states associated to graphs which encode their entanglement)

Thank you !

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