Positivity in Quantum Information Theory

Ion Nechita

CNRS and University of Ottawa

Toronto, August 4th 2011

Entanglement in Quantum Information Theory

- Quantum states with d degrees of freedom are described by density matrices

$$\rho \in \mathcal{M}^{1,+}(\mathbb{C}^d); \qquad \mathrm{Tr}\rho = 1 \text{ and } \rho \geqslant 0.$$

Entanglement in Quantum Information Theory

- Quantum states with d degrees of freedom are described by density matrices

$$\rho \in \mathcal{M}^{1,+}(\mathbb{C}^d); \qquad \mathrm{Tr}\rho = 1 \text{ and } \rho \geqslant 0.$$

- Two quantum systems

$$\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}).$$

Entanglement in Quantum Information Theory

- Quantum states with d degrees of freedom are described by density matrices

$$\rho \in \mathcal{M}^{1,+}(\mathbb{C}^d); \qquad \mathrm{Tr}\rho = 1 \text{ and } \rho \geqslant 0.$$

- Two quantum systems

$$\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}).$$

- A state ρ_{12} is called separable if it can be written as a convex combination of product states

$$\rho_{12} \in \mathcal{SEP} \iff \rho_{12} = \sum_i t_i \rho_1(i) \otimes \rho_2(i).$$

Entanglement in Quantum Information Theory

- Quantum states with d degrees of freedom are described by density matrices

$$\rho \in \mathcal{M}^{1,+}(\mathbb{C}^d); \qquad \mathrm{Tr}\rho = 1 \text{ and } \rho \geqslant 0.$$

- Two quantum systems

$$\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}).$$

- A state ρ_{12} is called separable if it can be written as a convex combination of product states

$$\rho_{12} \in \mathcal{SEP} \iff \rho_{12} = \sum_i t_i \rho_1(i) \otimes \rho_2(i).$$

- Equivalently,

$$\mathcal{SEP} = \mathrm{conv}\left[\mathcal{M}^{1,+}(\mathbb{C}^{d_1}) \otimes \mathcal{M}^{1,+}(\mathbb{C}^{d_2})\right].$$

Entanglement in Quantum Information Theory

- Quantum states with d degrees of freedom are described by density matrices

$$\rho \in \mathcal{M}^{1,+}(\mathbb{C}^d); \qquad \mathrm{Tr}\rho = 1 \text{ and } \rho \geqslant 0.$$

- Two quantum systems

$$\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}).$$

- A state ρ_{12} is called separable if it can be written as a convex combination of product states

$$\rho_{12} \in \mathcal{SEP} \iff \rho_{12} = \sum_i t_i \rho_1(i) \otimes \rho_2(i).$$

- Equivalently,

$$\mathcal{SEP} = \mathrm{conv}\left[\mathcal{M}^{1,+}(\mathbb{C}^{d_1}) \otimes \mathcal{M}^{1,+}(\mathbb{C}^{d_2})\right].$$

- Non-separable states are called entangled.

Entanglement in Quantum Information Theory

- Separable rank one states $\rho_{12} = P_{e \otimes f} = P_e \otimes P_f$.

Entanglement in Quantum Information Theory

- Separable rank one states $\rho_{12} = P_{e \otimes f} = P_e \otimes P_f$.
- Bell state or maximally entangled state $\rho_{12} = P_{\mathrm{Bell}}$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni \mathrm{Bell} = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2) \neq x \otimes y.$$

Entanglement in Quantum Information Theory

- Separable rank one states $\rho_{12} = P_{e \otimes f} = P_e \otimes P_f$.
- Bell state or maximally entangled state $\rho_{12} = P_{\mathrm{Bell}}$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni \mathrm{Bell} = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2) \neq x \otimes y.$$

- Separability / entanglement are defined wrt a fixed partition $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. States separable wrt any partition are called absolutely separable

$$\mathcal{ASEP} = \bigcap_{U \in \mathcal{U}_d} U \cdot \mathcal{SEP} \cdot U^*.$$

Entanglement in Quantum Information Theory

- Separable rank one states $\rho_{12} = P_{e \otimes f} = P_e \otimes P_f$.
- Bell state or maximally entangled state $\rho_{12} = P_{\mathrm{Bell}}$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni \mathrm{Bell} = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2) \neq x \otimes y.$$

- Separability / entanglement are defined wrt a fixed partition $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. States separable wrt any partition are called absolutely separable

$$\mathcal{ASEP} = \bigcap_{U \in \mathcal{U}_d} U \cdot \mathcal{SEP} \cdot U^*.$$

- Separability of ρ_{12} depends on the eigenvalues and eigenvectors of ρ_{12}; absolute separability depends only on the spectrum of ρ_{12}.

Deciding [absolute] separability

- "I would not call entanglement one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought." [Schrödinger]

Deciding [absolute] separability

- "I would not call entanglement one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought." [Schrödinger]
- Entanglement is essential to the exponential speed-up of some quantum algorithms.

Deciding [absolute] separability

- "I would not call entanglement one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought." [Schrödinger]
- Entan … tum algori …

Shor's quantum factoring algorithm

- Runs on a quantum computer with polynomial time $O(\log^3 N)$.
- Classical sieve algorithms run in sub-exponential time $O(\exp(\log^{1/3} N))$.
- Entanglement is necessary for the exponential speed-up.
- State of the art factorization : $15 = 3 \times 5$.

Deciding [absolute] separability

- "I would not call entanglement one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought." [Schrödinger]
- Entanglement is essential to the exponential speed-up of some quantum algorithms.
- Deciding if a given $\rho_{12} \in \mathcal{SEP}$ is NP-hard [Gurvitz].

Deciding [absolute] separability

- "I would not call entanglement one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought." [Schrödinger]
- Entanglement is essential to the exponential speed-up of some quantum algorithms.
- Deciding if a given $\rho_{12} \in \mathcal{SEP}$ is NP-hard [Gurvitz].
- There exist a finite number of inequalities that are equivalent to absolute separability [Hildebrand]; however, their number grows exponentially with the dimension.

Deciding [absolute] separability

- "I would not call entanglement one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought." [Schrödinger]
- Entanglement is essential to the exponential speed-up of some quantum algorithms.
- Deciding if a given $\rho_{12} \in \mathcal{SEP}$ is NP-hard [Gurvitz].
- There exist a finite number of inequalities that are equivalent to absolute separability [Hildebrand]; however, their number grows exponentially with the dimension.
- For rank one quantum states, entanglement can be detected and quantified by the von Neumann entropy

$$H(P_x) = S(\mathrm{sv}(x)) = -\sum_i s_i(x) \log s_i(x), \qquad x \in \mathbb{C}^{d_1} \otimes C^{d_2} \cong \mathcal{M}_{d_1 \times d_2}(\mathbb{C}).$$

Deciding [absolute] separability

- "I would not call entanglement one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought." [Schrödinger]
- Entanglement is essential to the exponential speed-up of some quantum algorithms.
- Deciding if a given $\rho_{12} \in \mathcal{SEP}$ is NP-hard [Gurvitz].
- There exist a finite number of inequalities that are equivalent to absolute separability [Hildebrand]; however, their number grows exponentially with the dimension.
- For rank one quantum states, entanglement can be detected and quantified by the von Neumann entropy

$$H(P_x) = S(\mathrm{sv}(x)) = -\sum_i s_i(x) \log s_i(x), \qquad x \in \mathbb{C}^{d_1} \otimes C^{d_2} \cong \mathcal{M}_{d_1 \times d_2}(\mathbb{C}).$$

- Detecting entanglement in $\mathbb{C}^2 \otimes C^2$ and $\mathbb{C}^2 \otimes C^3$ is trivial via the PPT criterion [Horodecki].

Positive Partial Transpose matrices

- A map $f : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^d)$ is called
  - positive if $A \geqslant 0 \implies f(A) \geqslant 0$;
  - completely positive if $f \otimes \mathrm{id}_k$ is positive for all $k \geqslant 1$.

Positive Partial Transpose matrices

- A map $f : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^d)$ is called
  - positive if $A \geqslant 0 \implies f(A) \geqslant 0$;
  - completely positive if $f \otimes \mathrm{id}_k$ is positive for all $k \geqslant 1$.
- Let $f : \mathcal{M}(\mathbb{C}^{d_1}) \to \mathcal{M}(\mathbb{C}^{d_1})$ be a completely positive map. Then, For every separable state $\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$, one has $f \otimes \mathrm{id}_{d_2}(\rho_{12}) \geqslant 0$.

Positive Partial Transpose matrices

- A map $f : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^d)$ is called
  - positive if $A \geqslant 0 \implies f(A) \geqslant 0$;
  - completely positive if $f \otimes \mathrm{id}_k$ is positive for all $k \geqslant 1$.
- Let $f : \mathcal{M}(\mathbb{C}^{d_1}) \to \mathcal{M}(\mathbb{C}^{d_1})$ be a completely positive map. Then, For every separable state $\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$, one has $f \otimes \mathrm{id}_{d_2}(\rho_{12}) \geqslant 0$.
- Let $f : \mathcal{M}(\mathbb{C}^{d_1}) \to \mathcal{M}(\mathbb{C}^{d_1})$ be a positive map. Then, for every separable state $\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$, one has $f \otimes \mathrm{id}_{d_2}(\rho_{12}) \geqslant 0$. Moreover, if there exists an entangled state σ_{12} such that $f \otimes \mathrm{id}_{d_2}(\sigma_{12}) \not\leqslant 0$, f is called an entanglement witness.

Positive Partial Transpose matrices

- A map $f : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^d)$ is called
  - positive if $A \geqslant 0 \implies f(A) \geqslant 0$;
  - completely positive if $f \otimes \mathrm{id}_k$ is positive for all $k \geqslant 1$.
- Let $f : \mathcal{M}(\mathbb{C}^{d_1}) \to \mathcal{M}(\mathbb{C}^{d_1})$ be a completely positive map. Then, For every separable state $\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$, one has $f \otimes \mathrm{id}_{d_2}(\rho_{12}) \geqslant 0$.
- Let $f : \mathcal{M}(\mathbb{C}^{d_1}) \to \mathcal{M}(\mathbb{C}^{d_1})$ be a positive map. Then, for every separable state $\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$, one has $f \otimes \mathrm{id}_{d_2}(\rho_{12}) \geqslant 0$. Moreover, if there exists an entangled state σ_{12} such that $f \otimes \mathrm{id}_{d_2}(\sigma_{12}) \not\leqslant 0$, f is called an entanglement witness.
- The transposition map t is an entanglement witness. Define the convex set

$$\mathcal{PPT} = \{\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \mid \mathrm{t}_{d_1} \otimes \mathrm{id}_{d_2}(\rho_{12}) \geqslant 0\}.$$

Positive Partial Transpose matrices

- A map $f : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^d)$ is called
  - positive if $A \geqslant 0 \implies f(A) \geqslant 0$;
  - completely positive if $f \otimes \mathrm{id}_k$ is positive for all $k \geqslant 1$.
- Let $f : \mathcal{M}(\mathbb{C}^{d_1}) \to \mathcal{M}(\mathbb{C}^{d_1})$ be a completely positive map. Then, For every separable state $\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$, one has $f \otimes \mathrm{id}_{d_2}(\rho_{12}) \geqslant 0$.
- Let $f : \mathcal{M}(\mathbb{C}^{d_1}) \to \mathcal{M}(\mathbb{C}^{d_1})$ be a positive map. Then, for every separable state $\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$, one has $f \otimes \mathrm{id}_{d_2}(\rho_{12}) \geqslant 0$. Moreover, if there exists an entangled state σ_{12} such that $f \otimes \mathrm{id}_{d_2}(\sigma_{12}) \not\leqslant 0$, f is called an entanglement witness.
- The transposition map t is an entanglement witness. Define the convex set

$$\mathcal{PPT} = \{\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \,|\, \mathrm{t}_{d_1} \otimes \mathrm{id}_{d_2}(\rho_{12}) \geqslant 0\}.$$

- For $(d_1, d_2) \in \{(2,2),(2,3)\}$ we have $\mathcal{SEP} = \mathcal{PPT}$. In other dimensions, the inclusion $\mathcal{SEP} \subset \mathcal{PPT}$ is strict.

The PPT criterion at work

- Recall the Bell state $\rho_{12} = P_{\mathrm{Bell}}$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni \mathrm{Bell} = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2).$$

The PPT criterion at work

- Recall the Bell state $\rho_{12} = P_{\mathrm{Bell}}$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni \mathrm{Bell} = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2).$$

- Written as a matrix in $\mathcal{M}^{1,+}(\mathbb{C}^4)$

$$\rho_{12} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

The PPT criterion at work

- Recall the Bell state $\rho_{12} = P_{\mathrm{Bell}}$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni \mathrm{Bell} = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2).$$

- Written as a matrix in $\mathcal{M}^{1,+}(\mathbb{C}^4)$

$$\rho_{12} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

- Partial transposition: transpose the block matrix $B = (B_{ij})$, keeping the blocks intact:

$$\rho_{12}^{\Gamma} = \mathrm{t}_2 \otimes \mathrm{id}_2(\rho_{12}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The PPT criterion at work

- Recall the Bell state $\rho_{12} = P_{\mathrm{Bell}}$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni \mathrm{Bell} = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2).$$

- Written as a matrix in $\mathcal{M}^{1,+}(\mathbb{C}^4)$

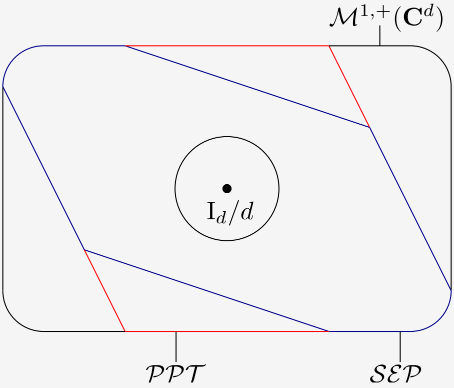
$$\rho_{12} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

- Partial transposition: transpose the block matrix $B = (B_{ij})$, keeping the blocks intact:

$$\rho_{12}^{\Gamma} = \mathrm{t}_2 \otimes \mathrm{id}_2(\rho_{12}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- This matrix is no longer positive $\Longrightarrow$ the state is entangled.

A family of convex sets



- States in $\mathcal{PPT} \setminus \mathcal{SEP}$ are called bound entangled: no "maximal" entangled can be distilled from them.
- All these sets contain an open ball around the identity.

Induced measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- In order to compare the volumes of the sets $[\mathcal{A}]\mathcal{SEP}$, $[\mathcal{A}]\mathcal{PPT}$, one needs a probability measure on the compact set $\mathcal{M}^{1,+}(\mathbb{C}^d)$ of positive, unit trace matrices.

Induced measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- In order to compare the volumes of the sets $[\mathcal{A}]\mathcal{SEP}$, $[\mathcal{A}]\mathcal{PPT}$, one needs a probability measure on the compact set $\mathcal{M}^{1,+}(\mathbb{C}^d)$ of positive, unit trace matrices.
- Let $X \in \mathcal{M}_{d\times s}(\mathbb{C})$ a rectangular $d \times s$ matrix with i.i.d. complex standard Gaussian entries. Define the random variables

$$W_{d,s} = XX^* \text{ and } \mathcal{M}^{1,+}(\mathbb{C}^d) \ni \rho_{d,s} = \frac{XX^*}{\mathrm{Tr}(XX^*)} = \frac{W_{d,s}}{\mathrm{Tr}\, W_{d,s}}.$$

Induced measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- In order to compare the volumes of the sets $[\mathcal{A}]\mathcal{SEP}$, $[\mathcal{A}]\mathcal{PPT}$, one needs a probability measure on the compact set $\mathcal{M}^{1,+}(\mathbb{C}^d)$ of positive, unit trace matrices.
- Let $X \in \mathcal{M}_{d\times s}(\mathbb{C})$ a rectangular $d \times s$ matrix with i.i.d. complex standard Gaussian entries. Define the random variables

$$W_{d,s} = XX^* \text{ and } \mathcal{M}^{1,+}(\mathbb{C}^d) \ni \rho_{d,s} = \frac{XX^*}{\operatorname{Tr}(XX^*)} = \frac{W_{d,s}}{\operatorname{Tr} W_{d,s}}.$$

- The random matrix $W_{d,s}$ is called a Wishart matrix and the distribution of $\rho_{d,s}$ is called the induced measure of parameters (d, s) and is noted $\mu_{d,s}$.

Induced measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- In order to compare the volumes of the sets $[\mathcal{A}]\mathcal{SEP}$, $[\mathcal{A}]\mathcal{PPT}$, one needs a probability measure on the compact set $\mathcal{M}^{1,+}(\mathbb{C}^d)$ of positive, unit trace matrices.
- Let $X \in \mathcal{M}_{d\times s}(\mathbb{C})$ a rectangular $d \times s$ matrix with i.i.d. complex standard Gaussian entries. Define the random variables

$$W_{d,s} = XX^* \text{ and } \mathcal{M}^{1,+}(\mathbb{C}^d) \ni \rho_{d,s} = \frac{XX^*}{\mathrm{Tr}(XX^*)} = \frac{W_{d,s}}{\mathrm{Tr}\, W_{d,s}}.$$

- The random matrix $W_{d,s}$ is called a Wishart matrix and the distribution of $\rho_{d,s}$ is called the induced measure of parameters (d, s) and is noted $\mu_{d,s}$.
- Almost surely, $\rho_{d,s}$ has full rank iff $s \geqslant d$.

Induced measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- In order to compare the volumes of the sets $[\mathcal{A}]\mathcal{SEP}$, $[\mathcal{A}]\mathcal{PPT}$, one needs a probability measure on the compact set $\mathcal{M}^{1,+}(\mathbb{C}^d)$ of positive, unit trace matrices.
- Let $X \in \mathcal{M}_{d\times s}(\mathbb{C})$ a rectangular $d \times s$ matrix with i.i.d. complex standard Gaussian entries. Define the random variables

$$W_{d,s} = XX^* \text{ and } \mathcal{M}^{1,+}(\mathbb{C}^d) \ni \rho_{d,s} = \frac{XX^*}{\mathrm{Tr}(XX^*)} = \frac{W_{d,s}}{\mathrm{Tr} W_{d,s}}.$$

- The random matrix $W_{d,s}$ is called a Wishart matrix and the distribution of $\rho_{d,s}$ is called the induced measure of parameters (d, s) and is noted $\mu_{d,s}$.
- Almost surely, $\rho_{d,s}$ has full rank iff $s \geqslant d$.
- The measure $\mu_{d,s}$ is unitarily invariant: there exist a probability measure $\nu_{d,s}$ on the probability simples $\Delta_d = \{\lambda \in \mathbb{R}^d \mid \lambda_i \geqslant 0, \sum \lambda_i = 1\}$ such that if $\lambda \sim \nu_{d,s}$ and U is a Haar unitary matrix independent of λ,

$$U\mathrm{diag}(\lambda)U^* \sim \mu_{d,s}.$$

Induced measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- In order to compare the volumes of the sets $[\mathcal{A}]\mathcal{SEP}$, $[\mathcal{A}]\mathcal{PPT}$, one needs a probability measure on the compact set $\mathcal{M}^{1,+}(\mathbb{C}^d)$ of positive, unit trace matri
- Let X … andard Gauss …
- The r … tion of $\rho_{d,s}$ i … $\mu_{d,s}$.
- Almo
- The r … sure $\nu_{d,s}$ on th … hat if $\lambda \sim \nu$

$$U\mathrm{diag}(\lambda)U^* \sim \mu_{d,s}.$$

Eigenvalues for induced measures

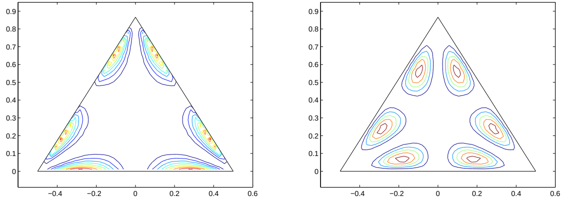


Figure: Induced measure eigenvalue distribution for $(d = 3, s = 3)$ and $(d = 3, s = 5)$.

Volume of convex sets under the induced measure

- Let $C \subset \mathcal{M}^{1,+}(\mathbb{C}^d)$ a convex body, with $\mathrm{I}_d/d \in C^\circ$. Then

$$\lim_{s\to\infty} \mu_{d,s}(C) = 1.$$

Volume of convex sets under the induced measure

- Let $C \subset \mathcal{M}^{1,+}(\mathbb{C}^d)$ a convex body, with $\mathrm{I}_d/d \in C^\circ$. Then

$$\lim_{s \to \infty} \mu_{d,s}(C) = 1.$$

Definition

A pair of functions $(s_0(d), s_1(d))$ are called a threshold for a family of convex sets $\{C_d\}_{d \geqslant 2}$ if both conditions below hold

- If $s \lesssim s_0(d)$, then

$$\lim_{s \to \infty} \mu_{d,s}(C_d) = 0;$$

- If $s \gtrsim s_1(d)$, then

$$\lim_{s \to \infty} \mu_{d,s}(C_d) = 1.$$

Threshold values for $[\mathcal{A}]\mathcal{SEP}$, $[\mathcal{A}]\mathcal{PPT}$

Set	Balanced $\mathbb{C}^d = \mathbb{C}^{\sqrt{d}} \otimes C^{\sqrt{d}}$	Unbalanced $\mathbb{C}^d = \mathbb{C}^p \otimes C^{d/p}$
$\mathcal{SEP}$	$s_0 = Cd^{3/2}$ $s_1 = Cd^{3/2} \log^2 d$ [Aubrun, Szarek, Ye]	$s_0 = Cpd$ $s_1 = Cpd \log^2 d$ [Aubrun, Szarek, Ye]
$\mathcal{PPT}$	$s_0 = s_1 = 4d$ [Aubrun]	$s_0 = \left[2 + 2\sqrt{1 - p^{-2}}\right] d$ Most likely $s_1 = s_0$ [Banica, N.]
$\mathcal{ASEP}$	? [Collins, N., Ye, in progress]	? [Collins, N., Ye, in progress]
$\mathcal{APPT}$	$s_0 = 64/(9\pi^2)d^2$ $s_1 = 4d^2$ [Collins, N., Ye]	$s_0 = s_1 = \left[p + \sqrt{p^2 - 1}\right]^2 d$ [Collins, N., Ye]

$\mathcal{PPT}$, unbalanced case

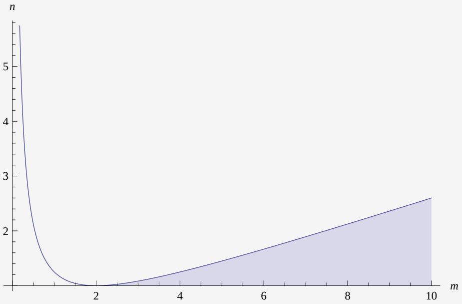
Theorem (Banica, N. - arXiv:1105.2556)

Let W be a complex Wishart matrix of parameters (dn, dm). Then, with $d \to \infty$, the empirical spectral distribution of mW^Γ converges in moments to a free difference of free Poisson distributions of respective parameters $m(n \pm 1)/2$.

Corollary

The limiting measure in the previous theorem has positive support iff

$$n \leqslant \frac{m}{4} + \frac{1}{m} \qquad \text{and} \qquad m \geqslant 2.$$



$\mathcal{PPT}$, unbalanced case

What's a free difference of free Poisson distributions ?

- The free Poisson distribution of parameter $c > 0$:

$$\pi_c = \max(1-c, 0)\delta_0 + \frac{\sqrt{4c - (x-1-c)^2}}{2\pi x} \mathbf{1}_{[1+c-2\sqrt{c}, 1+c+2\sqrt{c}]}(x)\, dx.$$

- Free additive convolution of two compactly supported probability distributions $\mu_{1,2}$: sample $X_{1,2} \in \mathbb{R}^n$ from $\mu_{1,2}$ and consider

$$A = U_1 \mathrm{diag}(X_1) U_1^* + U_2 \mathrm{diag}(X_2) U_2^*,$$

where $U_{1,2}$ are $n \times n$ independent Haar unitary matrices. Then, almost surely when $n \to \infty$, the spectrum of A has distribution $\mu_1 \boxplus \mu_2$.

$\mathcal{APPT}$, unbalanced case

Theorem (Collins, N., Ye - soon on arXiv)

Let ρ be a random quantum state from the induced ensemble of parameters (d, s). Almost surely, when $d \to \infty$ and $s \sim cd$, one has:

- *If $c > (p + \sqrt{p^2 - 1})^2$, then ρ is $\mathcal{APPT}$.*
- *Conversely, if $c < (p + \sqrt{p^2 - 1})^2$, then ρ is not $\mathcal{APPT}$.*

Theorem (Collins, N., Ye - soon on arXiv)

Let ρ be a random quantum state from the induced ensemble of parameters (d, s). Then, for all $\varepsilon > 0$, almost surely, when $d \to \infty$ and $s > (4 + \varepsilon)p^2 d$, the quantum state ρ is $\mathcal{APPT}$. The following converse holds:

- *When $1 \ll p^2 \ll d$ and $s < (4 - \varepsilon)p^2 d$, ρ is not $\mathcal{APPT}$.*
- *When $p^2 \sim \tau d$ for a constant $\tau \in (0, 1]$, there exists an explicitly computable constant C_τ such that whenever and $s < 4(C_\tau - \varepsilon)p^2 d$, ρ is not $\mathcal{APPT}$.*

Concluding remarks

- Detecting and quantifying entanglement is important in Quantum Information Theory.

Concluding remarks

- Detecting and quantifying entanglement is important in Quantum Information Theory.
- Checking separability is NP-hard, so easier criteria are needed.

Concluding remarks

- Detecting and quantifying entanglement is important in Quantum Information Theory.
- Checking separability is NP-hard, so easier criteria are needed.
- The PPT criterion works for 2×2 and 2×3 systems, but fails in general for larger ones.

Concluding remarks

- Detecting and quantifying entanglement is important in Quantum Information Theory.
- Checking separability is NP-hard, so easier criteria are needed.
- The PPT criterion works for 2×2 and 2×3 systems, but fails in general for larger ones.
- Relative volumes of $\mathcal{SEP}$ and $\mathcal{PPT}$ are a good indication of how well the PPT criterion performs.

Concluding remarks

- Detecting and quantifying entanglement is important in Quantum Information Theory.
- Checking separability is NP-hard, so easier criteria are needed.
- The PPT criterion works for 2×2 and 2×3 systems, but fails in general for larger ones.
- Relative volumes of $\mathcal{SEP}$ and $\mathcal{PPT}$ are a good indication of how well the PPT criterion performs.
- Random matrix theory and free probability are the right tools to tackle this problem.

Concluding remarks

- Detecting and quantifying entanglement is important in Quantum Information Theory.
- Checking separability is NP-hard, so easier criteria are needed.
- The PPT criterion works for 2×2 and 2×3 systems, but fails in general for larger ones.
- Relative volumes of $\mathcal{SEP}$ and $\mathcal{PPT}$ are a good indication of how well the PPT criterion performs.
- Random matrix theory and free probability are the right tools to tackle this problem.
- The "absolute" (basis independent) formulation of the problem are considered.

Merci !