Block-modified Wishart matrices and applications to entanglement theory

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joint work with Teodor Banica (Cergy)

Bedlewo, September 29, 2011

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- Non-separable states are called entangled.

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- For rank one quantum states, entanglement can be detected and quantified by the von Neumann entropy

$$H(P_x) = S(\mathrm{sv}(x)) = -\sum_i s_i(x) \log s_i(x), x \in \mathbb{C}^{d_1} \otimes C^{d_2} \cong \mathcal{M}_{d_1 \times d_2}(\mathbb{C}).$$

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- Detecting entanglement for general states $\mathbb{C}^2 \otimes C^2$ and $\mathbb{C}^2 \otimes C^3$ is trivial via the PPT criterion [Horodecki].

Positive Partial Transpose matrices

- A map $f : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^d)$ is called
  - positive if $A \geqslant 0 \implies f(A) \geqslant 0$;
  - completely positive if $\mathrm{id}_k \otimes f$ is positive for all $k \geqslant 1$.

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- Let $f : \mathcal{M}(\mathbb{C}^{d_2}) \to \mathcal{M}(\mathbb{C}^{d_2})$ be a completely positive map. Then, For every state $\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$, one has $[\mathrm{id}_{d_1} \otimes f](\rho_{12}) \geqslant 0$.

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- Let $f : \mathcal{M}(\mathbb{C}^{d_2}) \to \mathcal{M}(\mathbb{C}^{d_2})$ be a positive map. Then, for every separable state $\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$, one has $[\mathrm{id}_{d_1} \otimes f](\rho_{12}) \geqslant 0$.

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- The transposition map t is positive, but not CP. Define the convex set

$$\mathcal{PPT} = \{\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \,|\, [\mathrm{id}_{d_1} \otimes \mathrm{t}_{d_2}](\rho_{12}) \geqslant 0\}.$$

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- For $(d_1, d_2) \in \{(2,2),(2,3)\}$ we have $\mathcal{SEP} = \mathcal{PPT}$. In other dimensions, the inclusion $\mathcal{SEP} \subset \mathcal{PPT}$ is strict.

The PPT criterion at work

- Recall the Bell state $\rho_{12} = P_{\text{Bell}}$, where

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- Written as a matrix in $\mathcal{M}^{1,+}(\mathbb{C}^4)$

$$\rho_{12} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

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- Partial transposition: transpose each block B_{ij}:

$$\rho_{12}^{\Gamma} = [\mathrm{id}_2 \otimes \mathrm{t}_2](\rho_{12}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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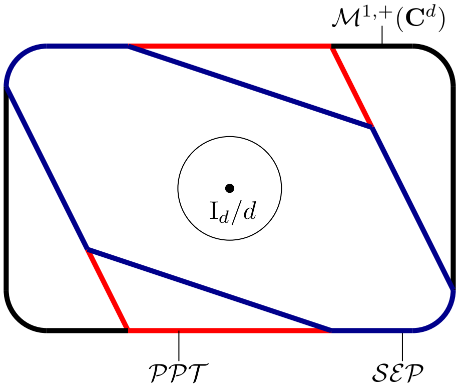
- This matrix is no longer positive $\implies$ the state is entangled.

Three convex sets



- States in $\mathcal{PPT} \setminus \mathcal{SEP}$ are called bound entangled: no “maximal” entangled can be distilled from them.
- All these sets contain an open ball around the identity.

The problem we consider

$$\mathcal{M}^{1,+}(\mathbb{C}^{d_1 d_2}) = \{\rho \mid \mathrm{Tr}\rho = 1 \text{ and } \rho \geqslant 0\}$$

$$\mathcal{SEP} = \left\{ \sum_i t_i \rho_1(i) \otimes \rho_2(i) \right\} = \mathrm{conv}\left[\mathcal{M}^{1,+}(\mathbb{C}^{d_1}) \otimes \mathcal{M}^{1,+}(\mathbb{C}^{d_2}) \right]$$

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Problem

Compare the convex sets

$$\mathcal{SEP} \subset \mathcal{PPT} \subset \mathcal{M}^{1,+}(\mathbb{C}^{d_1 d_2}).$$

Probability measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- Let $X \in \mathcal{M}_{d \times s}(\mathbb{C})$ a rectangular $d \times s$ matrix with i.i.d. complex standard Gaussian entries. Define the random variables

$$W_{d,s} = XX^* \text{ and } \mathcal{M}^{1,+}(\mathbb{C}^d) \ni \rho_{d,s} = \frac{XX^*}{\mathrm{Tr}(XX^*)} = \frac{W_{d,s}}{\mathrm{Tr}\, W_{d,s}}.$$

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- Almost surely, $\rho_{d,s}$ has full rank iff $s \geqslant d$.
- The measure $\mu_{d,s}$ is unitarily invariant: there exist a probability measure $\nu_{d,s}$ on the probability simples $\Delta_d = \{\lambda \in \mathbb{R}^d \mid \lambda_i \geqslant 0, \sum \lambda_i = 1\}$ such that if $\lambda \sim \nu_{d,s}$ and U is a Haar unitary matrix independent of λ,

$$U \mathrm{diag}(\lambda) U^* \sim \mu_{d,s}.$$

Eigenvalues for induced measures

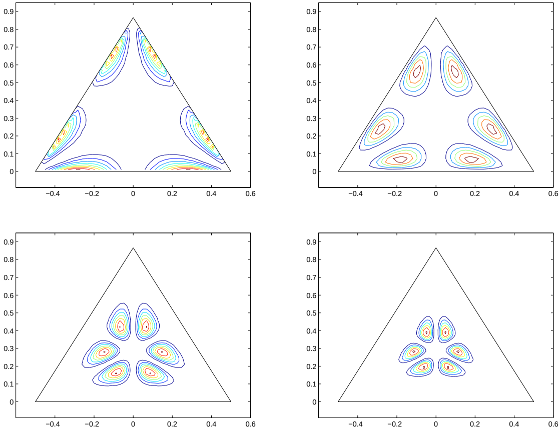

Figure: Induced measure eigenvalue distribution for $(d = 3, s = 3)$, $(d = 3, s = 5)$, $(d = 3, s = 7)$ and $(d = 3, s = 10)$.

Volume of convex sets under the induced measures

- Let $C \subset \mathcal{M}^{1,+}(\mathbb{C}^d)$ a convex body, with $\mathrm{I}_d/d \in C^\circ$. Then

$$\lim_{s\to\infty} \mu_{d,s}(C) = 1.$$

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Definition

A pair of functions $s_0(d), s_1(d)$ are called a threshold for a family of convex sets $\{C_d\}_{d\geqslant 2}$ if both conditions below hold

- If $s(d) \lesssim s_0(d)$, then

$$\lim_{d\to\infty} \mu_{d,s(d)}(C_d) = 0;$$

- If $s(d) \gtrsim s_1(d)$, then

$$\lim_{d\to\infty} \mu_{d,s(d)}(C_d) = 1.$$

Threshold for $\mathcal{SEP}$

Theorem (Aubrun, Szarek, Ye - 2011)

There exists a constant C such that the pair $s_0 = Cd^{3/2}$, $s_1 = Cd^{3/2}\log^2 d$ is a threshold for $\mathcal{SEP}$.
In other words, if $s < Cd^{3/2}$, then

$$\lim_{d\to\infty} \mu_{d,s}(\{\rho \text{ is entangled}\}) = 1$$

and if $s > Cd^{3/2}\log^2 d$, then

$$\lim_{d\to\infty} \mu_{d,s}(\{\rho \text{ is separable}\}) = 1.$$

Partial transposition of a Wishart matrix

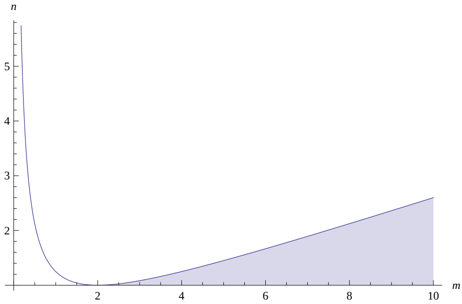
Theorem (Banica, N.)

Let W be a complex Wishart matrix of parameters (dn, dm). Then, with $d \to \infty$, the empirical spectral distribution of mW^Γ converges in moments to a free difference of free Poisson distributions of respective parameters $m(n \pm 1)/2$.

Corollary

The limiting measure in the previous theorem has positive support iff

$$n \leqslant \frac{m}{4} + \frac{1}{m} \text{ and } m \geqslant 2.$$



Threshold for $\mathcal{PPT}$, unbalanced & balanced case

Theorem (unbalanced case, Banica, N.)

In the unbalanced case $d_1 = d \to \infty$, $d_2 = n$ fixed, the lower bound of a threshold for $\mathcal{PPT}$ is given by $s_0 = \left[2 + 2\sqrt{1 - n^{-2}}\right] d$.

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Theorem (balanced case, Aubrun - 2010)

In the balanced case $d_1 = d_2 = d \to \infty$, a threshold pair for $\mathcal{PPT}$ is given by $s_0 = s_1 = 4d$.

Generalizing partial transposition

- Replace the transposition map t with an arbitrary, hermiticity preserving linear map $\varphi : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^d)$.

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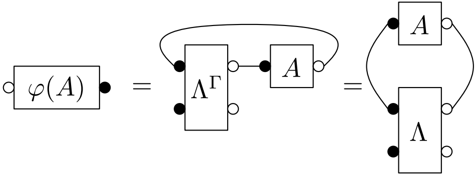

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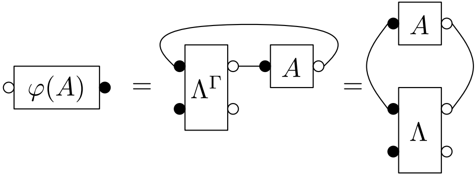


Problem

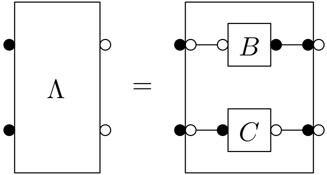
Compute the asymptotic spectrum of

$$\tilde{W} = (\mathrm{id} \otimes \varphi) W,$$

where W is a Wishart random matrix, $d \to \infty$ and n is fixed.

Some examples

- $\varphi(A) = Tr(BA)C$, in the case $C = c1$.
- $\Lambda = B^\top \otimes C$.



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- $\varphi(A) = BAC$, for any B, C.
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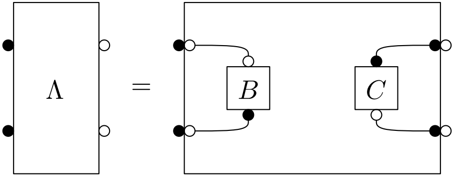

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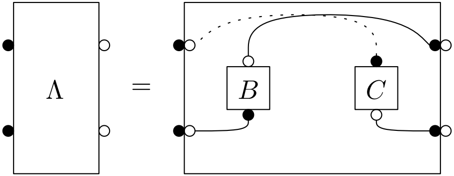
- $\Lambda = \mathrm{SWAP}_{BC}$,



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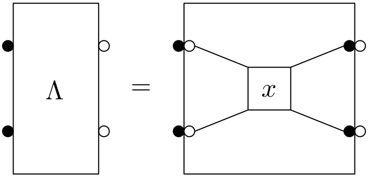
- $\varphi(A) = xA^{\delta}$, in the case $x = c1$.
- $\Lambda = \mathrm{Center}_x$,



Our result

Theorem (Banica, N. - work in progress)

Let $\tilde{W} = (id \otimes \varphi)W$, *where* W *is a complex Wishart matrix of parameters* (dn, dm), *and where* $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *is a self-adjoint linear map, coming from a matrix* $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. *Then, under suitable "planar" assumptions on* φ, *we have* $\delta m\tilde{W} \sim \pi_{mn\rho} \boxtimes \nu$, *with* $\rho = law(\Lambda)$, $\nu = law(D)$, $\delta = tr(D)$, *where* $D = \varphi(1)$

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- Identify the free cumulants, if the general term in the sum above is multiplicative.

Why does this fail for general φ ?

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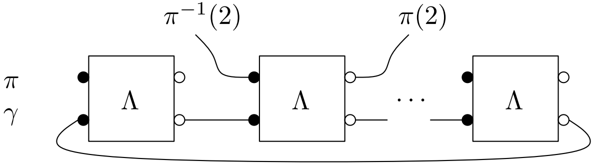

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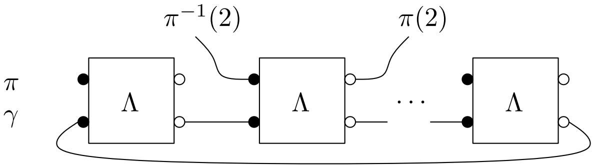

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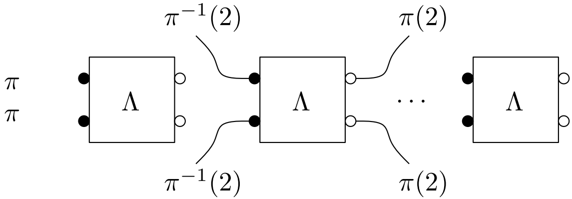

Thank you !

`http://arxiv.org/abs/1105.2556`

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work in progress