Block-modified Wishart matrices and applications to entanglement theory

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joint work with Teodor Banica (Cergy)

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Entanglement in Quantum Information Theory

- Quantum states with d degrees of freedom are described by density matrices

$$\rho \in \mathcal{M}^{1,+}(\mathbb{C}^d); \qquad \mathrm{Tr}\rho = 1 \text{ and } \rho \geqslant 0.$$

- Two quantum systems: $\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$.
- A state ρ_{12} is called separable if it can be written as a convex combination of product states

$$\rho_{12} \in \mathcal{SEP} \iff \rho_{12} = \sum_i t_i \rho_1(i) \otimes \rho_2(i),$$

  where $t_i \geqslant 0$, $\sum_i t_i = 1$, $\rho_1(i) \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1})$, $\rho_2(i) \in \mathcal{M}^{1,+}(\mathbb{C}^{d_2})$.
- Equivalently, $\mathcal{SEP} = \mathrm{conv}\left[\mathcal{M}^{1,+}(\mathbb{C}^{d_1}) \otimes \mathcal{M}^{1,+}(\mathbb{C}^{d_2})\right]$.
- Non-separable states are called entangled.

More on entanglement

- Deciding if a given ρ_{12} is separable is NP-hard [Gurvitz].
- For rank one quantum states, entanglement can be detected and quantified by the von Neumann entropy

$$H(P_x) = S(\mathrm{sv}(x)) = -\sum_i s_i(x) \log s_i(x), x \in \mathbb{C}^{d_1} \otimes C^{d_2} \cong \mathcal{M}_{d_1 \times d_2}(\mathbb{C}).$$

- Detecting entanglement for general states $\mathbb{C}^2 \otimes C^2$ and $\mathbb{C}^2 \otimes C^3$ is trivial via the PPT criterion [Horodecki].

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- A map $f : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^d)$ is called
  - positive if $A \geqslant 0 \implies f(A) \geqslant 0$;
  - completely positive if $\mathrm{id}_k \otimes f$ is positive for all $k \geqslant 1$.
- If $f : \mathcal{M}(\mathbb{C}^{d_2}) \to \mathcal{M}(\mathbb{C}^{d_2})$ is CP, then for every state ρ_{12} one has $[\mathrm{id}_{d_1} \otimes f](\rho_{12}) \geqslant 0$.
- If $f : \mathcal{M}(\mathbb{C}^{d_2}) \to \mathcal{M}(\mathbb{C}^{d_2})$ is only positive, then for every separable state ρ_{12}, one has $[\mathrm{id}_{d_1} \otimes f](\rho_{12}) \geqslant 0$.

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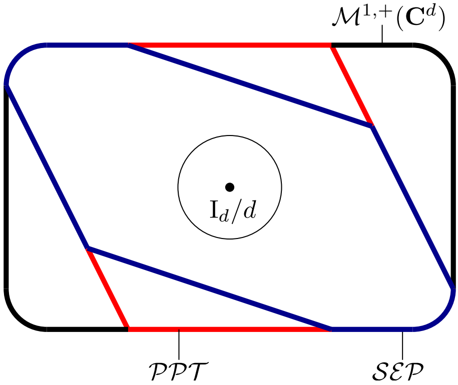
- The transposition map t is positive, but not CP. Put

$$\mathcal{PPT} = \{\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \mid [\mathrm{id}_{d_1} \otimes \mathrm{t}_{d_2}](\rho_{12}) \geqslant 0\}.$$

Three convex sets



- For $(d_1, d_2) \in \{(2,2), (2,3)\}$ we have $\mathcal{SEP} = \mathcal{PPT}$. In other dimensions, the inclusion $\mathcal{SEP} \subset \mathcal{PPT}$ is strict.
- States in $\mathcal{PPT} \setminus \mathcal{SEP}$ are called bound entangled: no "maximal" entangled can be distilled from them.
- All these sets contain an open ball around the identity.

The problem we consider

$$\mathcal{M}^{1,+}(\mathbb{C}^{d_1 d_2}) = \{\rho \,|\, \mathrm{Tr}\rho = 1 \text{ and } \rho \geqslant 0\}$$

$$\mathcal{SEP} = \left\{ \sum_i t_i \rho_1(i) \otimes \rho_2(i) \right\} = \mathrm{conv}\left[\mathcal{M}^{1,+}(\mathbb{C}^{d_1}) \otimes \mathcal{M}^{1,+}(\mathbb{C}^{d_2})\right]$$

$$\mathcal{PPT} = \{\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \,|\, [\mathrm{id}_{d_1} \otimes \mathrm{t}_{d_2}](\rho_{12}) \geqslant 0\}.$$

Problem

Compare the convex sets

$$\mathcal{SEP} \subset \mathcal{PPT} \subset \mathcal{M}^{1,+}(\mathbb{C}^{d_1 d_2}).$$

Probability measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- Let $X \in \mathcal{M}_{d\times s}(\mathbb{C})$ a rectangular $d \times s$ matrix with i.i.d. complex standard Gaussian entries. Define the random variables

$$W_{d,s} = XX^* \text{ and } \mathcal{M}^{1,+}(\mathbb{C}^d) \ni \rho_{d,s} = \frac{XX^*}{\mathrm{Tr}(XX^*)} = \frac{W_{d,s}}{\mathrm{Tr} W_{d,s}}.$$

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- The random matrix $W_{d,s}$ is called a Wishart matrix and the distribution of $\rho_{d,s}$ is called the induced measure of parameters (d,s) and is noted $\mu_{d,s}$.
- Almost surely, $\rho_{d,s}$ has full rank iff $s \geqslant d$.
- The measure $\mu_{d,s}$ is unitarily invariant: there exist a probability measure $\nu_{d,s}$ on the probability simples $\Delta_d = \{\lambda \in \mathbb{R}^d \mid \lambda_i \geqslant 0, \sum \lambda_i = 1\}$ such that if $\lambda \sim \nu_{d,s}$ and U is a Haar unitary matrix independent of λ,

$$U \mathrm{diag}(\lambda) U^* \sim \mu_{d,s}.$$

Eigenvalues for induced measures

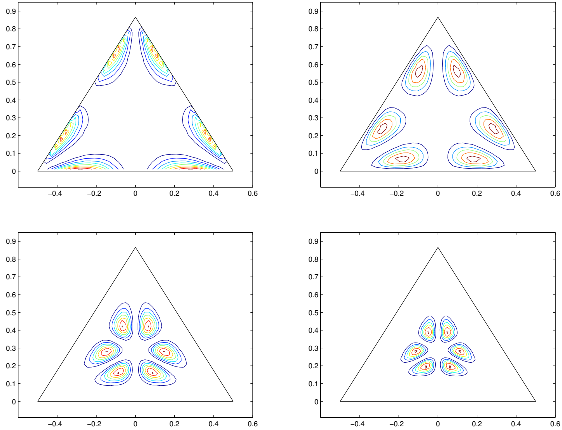

Figure: Induced measure eigenvalue distribution for $(d = 3, s = 3)$, $(d = 3, s = 5)$, $(d = 3, s = 7)$ and $(d = 3, s = 10)$.

Volume of convex sets under the induced measures

- Let $C \subset \mathcal{M}^{1,+}(\mathbb{C}^d)$ a convex body, with $\mathrm{I}_d/d \in C^\circ$. Then

$$\lim_{s\to\infty} \mu_{d,s}(C) = 1.$$

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Definition

A pair of functions $s_0(d), s_1(d)$ are called a threshold for a family of convex sets $\{C_d\}_{d\geqslant 2}$ if both conditions below hold

- If $s(d) \lesssim s_0(d)$, then

$$\lim_{d\to\infty} \mu_{d,s(d)}(C_d) = 0;$$

- If $s(d) \gtrsim s_1(d)$, then

$$\lim_{d\to\infty} \mu_{d,s(d)}(C_d) = 1.$$

Threshold for $\mathcal{SEP}$

Theorem (Aubrun, Szarek, Ye - 2011)

Guillaume's talk tomorrow

Partial transposition of a Wishart matrix

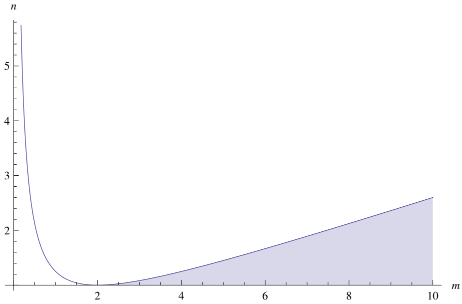
Theorem (Banica, N.)

Let W be a complex Wishart matrix of parameters (dn, dm). Then, with $d \to \infty$, the empirical spectral distribution of mW^Γ converges in moments to a free difference of free Poisson distributions of respective parameters $m(n \pm 1)/2$.

Corollary

The limiting measure in the previous theorem has positive support iff

$$n \leqslant \frac{m}{4} + \frac{1}{m} \text{ and } m \geqslant 2.$$



What is a free difference of free Poison measures ?

- Free additive convolution (or free sum) of two compactly supported probability distributions $\mu_{1,2}$: sample $X_{1,2} \in \mathbb{R}^n$ from $\mu_{1,2}$ and consider

$$A = U_1 \mathrm{diag}(X_1) U_1^* + U_2 \mathrm{diag}(X_2) U_2^*,$$

where $U_{1,2}$ are $n \times n$ independent Haar unitary rotations. Then, as $n \to \infty$, the spectrum of A has distribution $\mu_1 \boxplus \mu_2$.

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- The free Poisson distribution of parameter $c > 0$:

$$\pi_c = \max(1-c, 0)\delta_0 + \frac{\sqrt{4c - (x-1-c)^2}}{2\pi x} \mathbf{1}_{[(1-\sqrt{c})^2, (1+\sqrt{c})^2]}(x)\, dx.$$

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- Moreover, π_c is the limit eigenvalue distribution of a rescaled density matrix from the induced ensemble $\rho_{d,cd}$ (d large).

Threshold for $\mathcal{PPT}$, unbalanced & balanced case

Theorem (unbalanced case, Banica, N.)

In the unbalanced case $d_1 = d \to \infty$, $d_2 = n$ fixed, the lower bound of a threshold for $\mathcal{PPT}$ is given by $s_0 = \left[2 + 2\sqrt{1 - n^{-2}}\right] d$.

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Theorem (balanced case, Aubrun - 2010)

In the balanced case $d_1 = d_2 = d \to \infty$, a threshold pair for $\mathcal{PPT}$ is given by $s_0 = s_1 = 4d$.

Generalizing partial transposition

- Replace the transposition map t with an arbitrary, hermiticity preserving linear map $\varphi : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^d)$.

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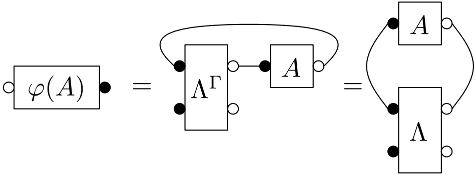
$$\varphi(A) = (\mathrm{Tr} \otimes id)[(\mathrm{t} \otimes \mathrm{id})\Lambda \cdot (A \otimes 1)]$$



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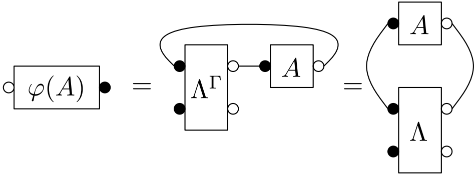


Problem

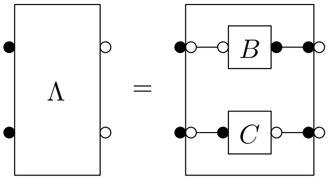
Compute the asymptotic spectrum of

$$\tilde{W} = (\mathrm{id} \otimes \varphi)W,$$

where W is a Wishart random matrix, $d \to \infty$ and n is fixed.

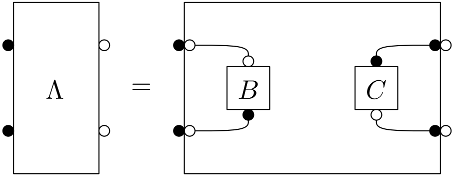
Some examples

- $\varphi(A) = Tr(BA)C$, in the case $C = c1$.
- $\Lambda = B^\top \otimes C$.



Some examples

- $\varphi(A) = BAC$, for any B, C.
- $\Lambda = |B\rangle\langle C|$,



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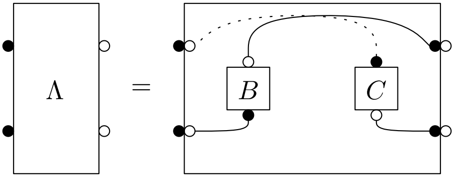
- $\varphi(A) = BA^t C$, in the case $BC = c1$.
- $\Lambda = \mathrm{SWAP}_{BC}$,



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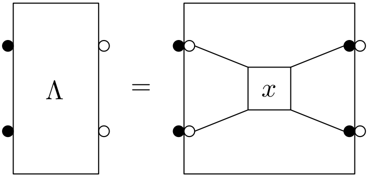
- $\varphi(A) = xA^{\delta}$, in the case $x = c1$.
- $\Lambda = \mathrm{Center}_x$,



Our result

Theorem (Banica, N. - work in progress)

Let $\tilde{W} = (id \otimes \varphi)W$, where W is a complex Wishart matrix of parameters (dn, dm), and where $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a self-adjoint linear map, coming from a matrix $\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. Then, under suitable "planar" assumptions on φ, we have $\delta m \tilde{W} \sim \pi_{mn\rho} \boxtimes \nu$, with $\rho = law(\Lambda)$, $\nu = law(D)$, $\delta = tr(D)$, where $D = \varphi(1)$

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- Idea of the proof

$$\lim_{d\to\infty} (\mathbb{E} \circ \mathrm{tr})((m\tilde{W})^p) = \sum_{\pi \in NC(p)} (mn)^{\#\pi} \mathrm{tr}_{(\pi,\gamma)}(\Lambda).$$

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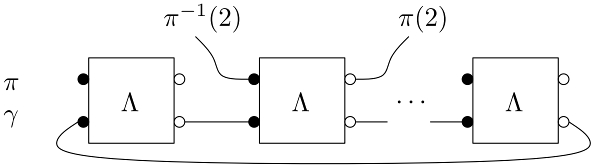
- Identify the free cumulants, if the general term in the sum above is multiplicative.

Why does this fail for general φ ?

- We have



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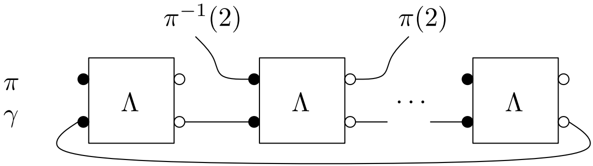

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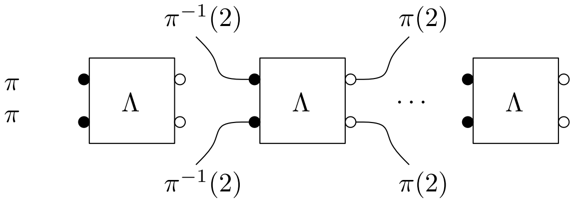

Thank you !

`http://arxiv.org/abs/1105.2556`

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