Random graph states, maximum flow on networks and the Fuss-Catalan ensembles of density matrices

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Random density matrices

— an overview —

Random pure states and the induced ensemble

- A random pure state in $\mathcal{H} = \mathbb{C}^N$ is a uniform point on the unit sphere of $\mathbb{C}^N$.
- One can sample from this distribution by normalizing a vector of N i.i.d. complex Gaussian random variables.
- Induced ensemble : partial trace a random pure state on a composite system $\mathcal{H} \otimes \mathcal{K}$:
$$\rho = \operatorname{Tr}_K |\psi\rangle\langle\psi|,$$
where $|\psi\rangle$ is a random pure state on $\mathbb{C}^N \otimes \mathbb{C}^K$.
- The random matrix ρ has the same distribution as a rescaled Wishart matrix $W / \operatorname{Tr} W$, where $W = XX^*$ with X a Ginibre matrix from $\mathcal{M}_{N \times K}(\mathbb{C})$.
- The eigenvalue density of ρ is given by
$$(\lambda_1, \ldots, \lambda_N) \mapsto C_{N,K} \exp\left(-\sum_{i=1}^{N} \lambda_i\right) \prod_{i=1}^{N} \lambda_i^{K-N} \Delta(\lambda)^2,$$
where $C_{N,K}$ is the constant $\left[\prod_{j=0}^{N-1} \Gamma(N+1-j)\Gamma(K-j)\right]^{-1}$ and
$$\Delta(\lambda) = \prod_{1 \leqslant i < j \leqslant N} (\lambda_i - \lambda_j).$$

Random density matrices - asymptotics

- In the limit $N \to \infty$, $K \sim cN$, for a fixed constant $c > 0$, the empirical spectral distribution of the rescaled eigenvalues

$$L_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{cN\lambda_i},$$

  converges almost surely to the Marchenko-Pastur distribution $\pi_c^{(1)}$.

- The Marchenko-Pastur (or free Poisson) distribution is defined by

$$\pi_c^{(1)} = \max\{1 - c, 0\}\delta_0 + \frac{\sqrt{(x - a)(b - x)}}{2\pi x} \mathbf{1}_{[a,b]}(x)dx,$$

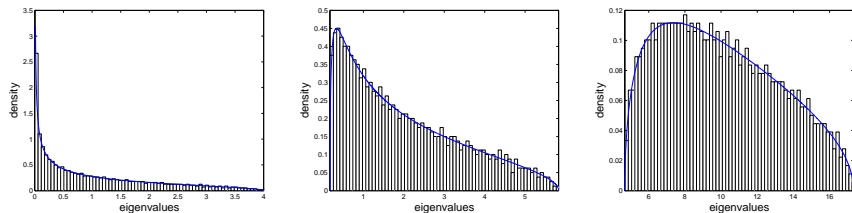 where $a = (\sqrt{c} - 1)^2$ and $b = (\sqrt{c} + 1)^2$.

Figure: Empirical and limit measures for $(N = 1000, K = 1000)$, $(N = 1000, K = 2000)$ and $(N = 1000, K = 10000)$.

Random graph states

— definition and examples —

Pure states associated to graphs

- Consider an undirected graph Γ consisting of m *edges* (or *bonds*) $B_1, \ldots, B_m$ and k *vertices* $V_1, \ldots V_k$.
- We associate to Γ a pure state $|\tilde{\Psi}\rangle\langle\tilde{\Psi}| \in \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{2m}$:

$$|\tilde{\Psi}\rangle = \bigotimes_{\{i,j\} \text{ edge}} |\Phi_{i,j}^+\rangle,$$

where $|\Phi_{i,j}^+\rangle$ denotes a maximally entangled state:

$$|\Phi_{ij}^+\rangle = \frac{1}{\sqrt{d_i N}} \sum_{x=1}^{d_i N} |e_x\rangle \otimes |f_x\rangle,$$

- $\dim \mathcal{H}_i = d_i N$, with d_i fixed parameters and $N \to \infty$. For each edge $\{i, j\}$, we have $d_i = d_j$.
- At each vertex, a Haar unitary matrix acts on the subsystems

$$\bigotimes_{i=1}^{n=2m} \mathcal{H}_i \ni |\Psi_\Gamma\rangle = \left(\bigotimes_{C \text{ vertex}} U_C \right) |\tilde{\Psi}\rangle$$

- The random unitary matrices $U_1, \ldots, U_k$ are independent.

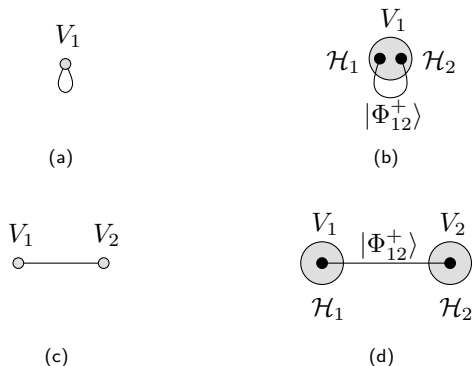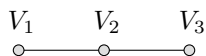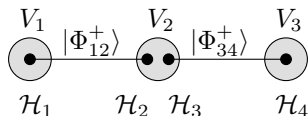# Pure states associated to graphs - examples

Figure: Graphs with one edge: a loop on one vertex, in simplified notation (a) and in the standard notation (b), and two vertices connected by one edge, in simplified notation (c) and in the standard notation (d).
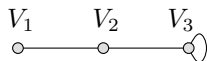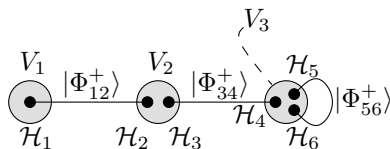
Figure: A linear 2-edge graph, in the simplified notation (a) and in the standard notation (b). Graph consisting of 3 vertices and 3 bonds (c), one of which is connected to the same vertex so it forms a loop; (d) the corresponding ensemble of random pure states defined in a Hilbert space composed of 6 subspaces represented by dark dots.

Marginals of graph states

— moments and entropy —

Partial tracing random pure graph states

- Non-local properties of the random graph state $|\Psi\rangle \rightsquigarrow$ partition of the set of all $2m$ subsystems into two groups, $\{S, T\}$.
- Total Hilbert space can be decomposed as a tensor product, $\mathcal{H} = \mathcal{H}_T \otimes \mathcal{H}_S$.
- Reduced density operator

$$\rho_S = \mathrm{Tr}_T \, |\Psi\rangle\langle\Psi|.$$

- Graphically, partial traces are denoted at the graph by "crossing" the spaces $\mathcal{H}_i$ which are being traced out.
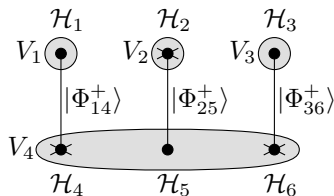

Figure: The random pure state supported on $n = 6$ subspaces is partial traced over the subspace $\mathcal{H}_T$ defined by the set $T = \{2, 4, 6\}$, represented by crosses. The reduced state ρ_S supported on subspaces corresponding to the set $S = \{1, 3, 5\}$.

Moments

- Use the method of moments: compute $\lim_{N \to \infty} \mathbb{E}\operatorname{Tr}(X^p)$ for a random matrix X.
- Using matrix coordinates, we can reduce our problem to computing integrals over the unitary group.

Theorem (Weingarten formula)

Let d be a positive integer and $\mathbf{i} = (i_1, \ldots, i_p)$, $\mathbf{i}' = (i'_1, \ldots, i'_p)$, $\mathbf{j} = (j_1, \ldots, j_p)$, $\mathbf{j}' = (j'_1, \ldots, j'_p)$ be p-tuples of positive integers from $\{1, 2, \ldots, d\}$. Then
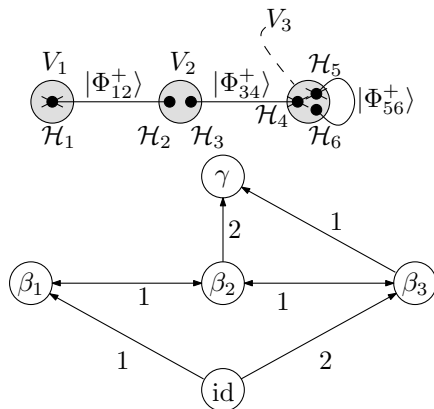
$$\int_{\mathcal{U}(d)} U_{i_1 j_1} \cdots U_{i_p j_p} \overline{U_{i'_1 j'_1}} \cdots \overline{U_{i'_p j'_p}} \, dU =$$

$$\sum_{\alpha, \beta \in \mathcal{S}_p} \delta_{i_1 i'_{\alpha(1)}} \ldots \delta_{i_p i'_{\alpha(p)}} \delta_{j_1 j'_{\beta(1)}} \ldots \delta_{j_p j'_{\beta(p)}} \operatorname{Wg}(d, \alpha\beta^{-1}).$$

If $p \neq p'$ then

$$\int_{\mathcal{U}(d)} U_{i_1 j_1} \cdots U_{i_p j_p} \overline{U_{i'_1 j'_1}} \cdots \overline{U_{i'_{p'} j'_{p'}}} \, dU = 0.$$

Network associated to a marginal

- Using the Weingarten formula, one has to find the dominating term in a sum indexed by permutations of p objects.
- This optimization problem is equivalent to finding the maximum flow in a network.

Network associated to a marginal

- Network $(\mathcal{V}, \mathcal{E}, w)$ with vertex set $\mathcal{V}$, edge set $\mathcal{E}$ and edge capacities w.
- The vertex set $\mathcal{V} = \{\text{id}, \gamma, \beta_1, \ldots, \beta_k\}$, with two distinguished vertices: the *source* id and the *sink* γ.
- The edges in $\mathcal{E}$ are oriented and they are of three types:

$$\mathcal{E} = \{(\text{id}, \beta_i) \; ; \; |T_i| > 0\} \sqcup \{(\beta_i, \gamma) \; ; \; |S_i| > 0\} \sqcup \{(\beta_i, \beta_j), (\beta_j, \beta_i) \; ; \; |E_{ij}| > 0\},$$

where S_i, T_i is are the surviving and traced out subsystems at vertex i and E_{ij} are the edges from vertex i to vertex j.
- The capacities of the edges are given by:

$$w(\text{id}, \beta_i) = |T_i| > 0$$
$$w(\beta_i, \gamma) = |S_i| > 0$$
$$w(\beta_i, \beta_j) = w(\beta_j, \beta_i) = |E_{ij}| > 0.$$

Main theorem

Theorem

Asymptotically, as $N \to \infty$, the p-th moment of the reduced density matrix behaves as

$$\mathbb{E} \operatorname{Tr}(\rho_S^p) \sim N^{-X(p-1)} \cdot [\text{ combinatorial term } + o(1)],$$

where X is the maximum flow in the network associated to the marginal. The combinatorial part can be expressed in terms of the residual network obtained after removing the capacities of the edges that appear in the maximum flow solution.
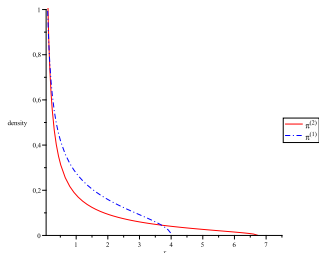
Fuss-Catalan limit distributions

Definition

- Matrix model: $\pi^{(s)}$ is the limit eigenvalue distribution of the random matrix $X_s = G_s \cdots G_2 G_1 G_1^* G_2^* \cdots G_s^*$, with i.i.d. Gaussian matrices G_i.
- Combinatorics: moments given by

$$\int x^p \, d\pi^{(s)}(x) = \frac{1}{sp+1} \binom{sp+p}{p}$$
$$= |\{\hat{0}_p \leqslant \sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_s \leqslant \hat{1}_p \in NC(p)\}|.$$

- Free probability: $\pi^{(s)} = \left(\pi^{(1)}\right)^{\boxtimes s}$, where $\pi^{(1)}$ is the free Poisson (or Marchenko-Pastur) distribution (of parameter $c = 1$).
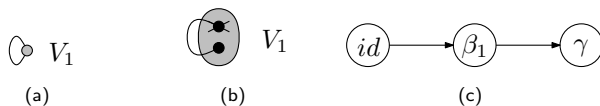
Figure: A vertex with one loop (a) and a marginal (b) having as a limit eigenvalue distribution the Marchenko-Pastur law $\pi^{(1)}$. In the network (c), both edges have capacity one.

- This is the simplest graph state having the Marchenko-Pastur asymptotic distribution.
- The reduced matrix is obtained by partial tracing an uniformly distributed pure state, hence it is an element of the *induced ensemble*.

Graph marginals with limit Fuss-Catalan distribution, $s = 2$

Figure: A graph (a) and a marginal (b) having as a limit eigenvalue distribution the Fuss-Catalan law $\pi^{(2)}$. In the network (c), non-labeled edges have capacity one. A maximum flow of 3 can be sent from the source id to the sink γ: one unit through each path id $\to \beta_i \to \gamma$, $i = 1, 2$ and one unit through the path id $\to \beta_1 \to \beta_2 \to \gamma$. In this way, the residual network is empty and the only constraint on the geodesic permutations β_1, β_2 is $\hat{0}_p \leqslant [\beta_1] \leqslant [\beta_2] \leqslant \hat{1}_p$, i.e. $[\beta_1]$ and $[\beta_2]$ form a 2-chain in $NC(p)$
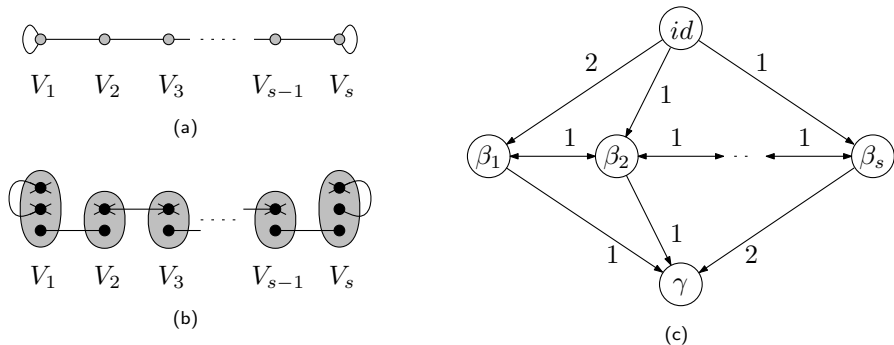
Figure: An example of a graph state (a) with a marginal (b) having as a limit eigenvalue distribution the s-th Fuss-Catalan probability measure $\pi^{(s)}$. The associated network (c) has a maximal flow of $s + 1$, obtained by sending a unit of flow through each β_i and a unit through the path $id \to \beta_1 \to \cdots \to \beta_s \to \gamma$. The linear chain condition $[\beta_1] \leqslant \cdots \leqslant [\beta_s]$ follows.

Area laws

Area law holds for adapted marginals of graph states

- Setting: quantum many-body problem with *local interactions*
- "Area law" : the entanglement entropy of ground states grows like the boundary area of the subregion
- Non-extensive behavior for the entanglement entropy.
- A marginal ρ_S is called *adapted* if the number of traced out systems in each vertex is either zero or maximal.

Theorem

Let ρ_S be an adapted marginal of a graph state $|\Psi\rangle\langle\Psi|$. Then

$$H(\rho_S) = |\partial S| \log N + O(1),$$

in the limit $N \to \infty$. The boundary ∂S contains all the edges between the "traced out" vertices and the "surviving" vertices.
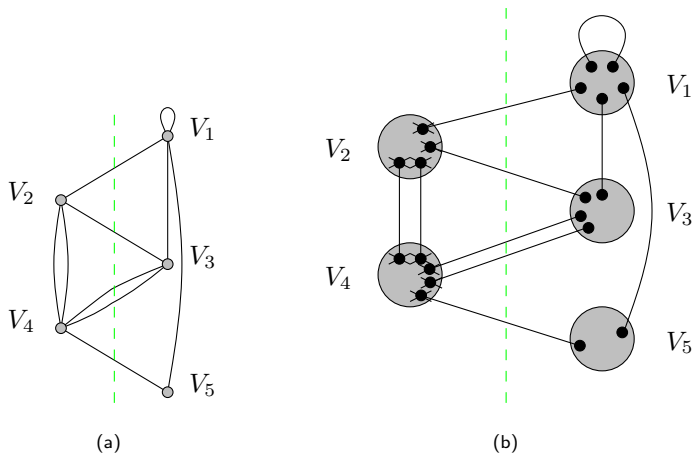
Figure: An example of a graph state (a) with an adapted marginal (b). The green dashed line represents the *boundary* between the traced and the surviving subsystems.

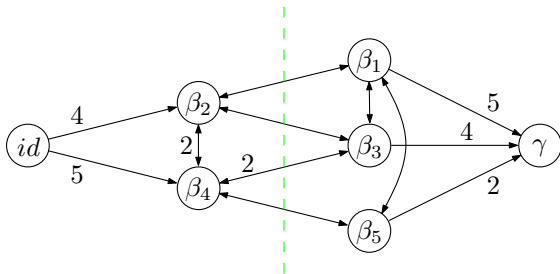# Area law holds for adapted marginals of graph states

Figure: The network associated to an adapted marginal. Nodes cannot be connected to both the source and the sink. The maximum flow equals the minimum cut in the network which is the number of edges in the boundary ∂S.

Conclusion and perspectives

Conclusion and perspectives

- Study lattice graphs.
- Other examples of physically and graph-theoretical motivated random states.
- Exotic limit distributions.
- Connection with free probability: classical and free multiplicative convolution semigroups.
- More general area laws.
- Dual graphs: vertices are GHZ states and edges represent unitary coupling.

Thank you !

http://arxiv.org/abs/1003.3075