Block-modified Wishart matrices and applications to entanglement theory

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GDR IQFA Workshop
Paris, IHP, November 23, 2011

Entanglement in Quantum Information Theory

- Quantum states with d degrees of freedom are described by density matrices

$$\rho \in \mathcal{M}^{1,+}(\mathbb{C}^d); \qquad \mathrm{Tr}\rho = 1 \text{ and } \rho \geqslant 0.$$

- Two quantum systems: $\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$.
- A state ρ_{12} is called separable if it can be written as a convex combination of product states

$$\rho_{12} \in \mathcal{SEP} \iff \rho_{12} = \sum_i t_i \rho_1(i) \otimes \rho_2(i),$$

  where $t_i \geqslant 0$, $\sum_i t_i = 1$, $\rho_1(i) \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1})$, $\rho_2(i) \in \mathcal{M}^{1,+}(\mathbb{C}^{d_2})$.
- Equivalently, $\mathcal{SEP} = \mathrm{conv}\left[\mathcal{M}^{1,+}(\mathbb{C}^{d_1}) \otimes \mathcal{M}^{1,+}(\mathbb{C}^{d_2})\right]$.
- Non-separable states are called entangled.

More on entanglement

- Deciding if a given general ρ_{12} is separable is NP-hard [Gurvitz].
- For rank one quantum states, entanglement can be detected and quantified by the von Neumann entropy

$$H(P_x) = S(\mathrm{sv}(x)) = -\sum_i s_i(x) \log s_i(x), x \in \mathbb{C}^{d_1} \otimes C^{d_2} \cong \mathcal{M}_{d_1 \times d_2}(\mathbb{C}).$$

- Separable rank one states

$$\rho_{12} = P_{e \otimes f} = P_e \otimes P_f .$$

- Detecting entanglement for general states $\mathbb{C}^2 \otimes C^2$ and $\mathbb{C}^2 \otimes C^3$ is trivial via the PPT criterion [Horodecki].

More on entanglement

- A map $f : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^d)$ is called
  - positive if $A \geqslant 0 \implies f(A) \geqslant 0$;
  - completely positive if $\mathrm{id}_k \otimes f$ is positive for all $k \geqslant 1$.

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- If $f : \mathcal{M}(\mathbb{C}^{d_2}) \to \mathcal{M}(\mathbb{C}^{d_2})$ is CP, then for every state ρ_{12} one has

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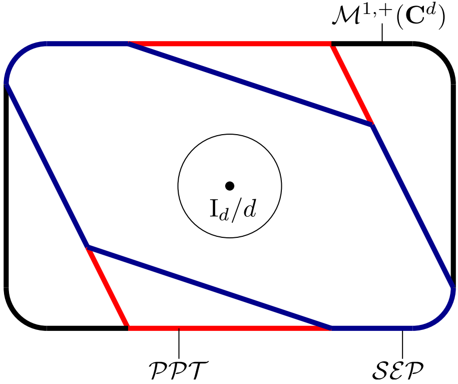
- The transposition map t is positive, but not CP. Put

$$\mathcal{PPT} = \{\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \,|\, [\mathrm{id}_{d_1} \otimes \mathrm{t}_{d_2}](\rho_{12}) \geqslant 0\}.$$

Three convex sets



- For $(d_1, d_2) \in \{(2,2),(2,3)\}$ we have $\mathcal{SEP} = \mathcal{PPT}$. In other dimensions, the inclusion $\mathcal{SEP} \subset \mathcal{PPT}$ is strict.
- States in $\mathcal{PPT} \setminus \mathcal{SEP}$ are called bound entangled: no "maximal" entangled can be distilled from them.
- All these sets contain an open ball around the identity.

The problem we consider

$$\mathcal{M}^{1,+}(\mathbb{C}^{d_1 d_2}) = \{\rho \,|\, \mathrm{Tr}\rho = 1 \text{ and } \rho \geqslant 0\}$$

$$\mathcal{SEP} = \left\{ \sum_i t_i \rho_1(i) \otimes \rho_2(i) \right\} = \mathrm{conv}\left[\mathcal{M}^{1,+}(\mathbb{C}^{d_1}) \otimes \mathcal{M}^{1,+}(\mathbb{C}^{d_2}) \right]$$

$$\mathcal{PPT} = \{\rho_{12} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) \,|\, [\mathrm{id}_{d_1} \otimes \mathrm{t}_{d_2}](\rho_{12}) \geqslant 0\}.$$

Problem

Compare the convex sets

$$\mathcal{SEP} \subset \mathcal{PPT} \subset \mathcal{M}^{1,+}(\mathbb{C}^{d_1 d_2}).$$

Probability measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- Let $X \in \mathcal{M}_{d\times s}(\mathbb{C})$ a rectangular $d \times s$ matrix with i.i.d. complex standard Gaussian entries. Define the random variables

$$W_{d,s} = XX^* \text{ and } \mathcal{M}^{1,+}(\mathbb{C}^d) \ni \rho_{d,s} = \frac{XX^*}{\mathrm{Tr}(XX^*)} = \frac{W_{d,s}}{\mathrm{Tr}\, W_{d,s}}.$$

- The random matrix $W_{d,s}$ is called a Wishart matrix and the distribution of $\rho_{d,s}$ is called the induced measure of parameters (d, s) and is noted $\mu_{d,s}$.

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- Open quantum systems point of view : let $x \in \mathbb{C}^d \otimes \mathbb{C}^s$ a unit norm vector (pure state).
- If x is distributed uniformly on the unit sphere of $\mathbb{C}^{ds}$, then its partial trace

$$\rho_{d,s} = \mathrm{Tr}_s P_x$$

has distribution $\mu_{d,s}$.

Eigenvalues for induced measures

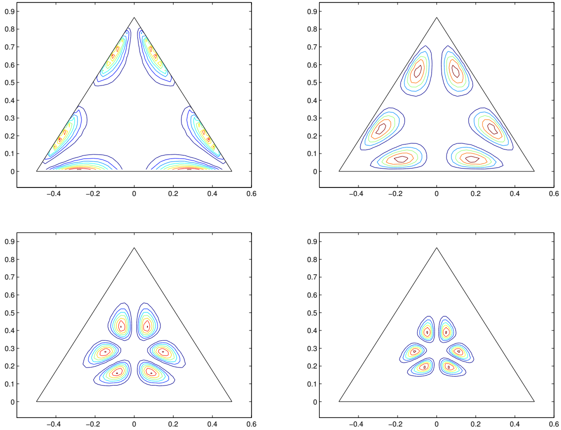

Figure: Induced measure eigenvalue distribution for $(d = 3, s = 3)$, $(d = 3, s = 5)$, $(d = 3, s = 7)$ and $(d = 3, s = 10)$.

Volume of convex sets under the induced measures

- Let $C \subset \mathcal{M}^{1,+}(\mathbb{C}^d)$ a convex body, with $\mathrm{I}_d/d \in C^\circ$. Then

$$\lim_{s\to\infty} \mu_{d,s}(C) = 1.$$

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Definition

A pair of functions $s_0(d), s_1(d)$ are called a threshold for a family of convex sets $\{C_d\}_{d\geqslant 2}$ if both conditions below hold

- If $s(d) \lesssim s_0(d)$, then

$$\lim_{d\to\infty} \mu_{d,s(d)}(C_d) = 0;$$

- If $s(d) \gtrsim s_1(d)$, then

$$\lim_{d\to\infty} \mu_{d,s(d)}(C_d) = 1.$$

Partial transposition of a Wishart matrix

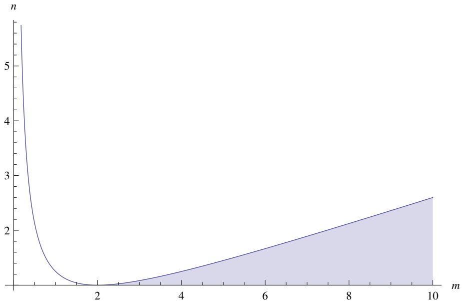
Theorem (Banica, N.)

Let W be a complex Wishart matrix of parameters (dn, dm). Then, with $d \to \infty$, the empirical spectral distribution of mW^Γ converges in moments to a free difference of free Poisson distributions of respective parameters $m(n \pm 1)/2$.

Corollary

The limiting measure in the previous theorem has positive support iff

$$n \leqslant \frac{m}{4} + \frac{1}{m} \text{ and } m \geqslant 2.$$



What is a free difference of free Poison measures ?

- Free additive convolution (or free sum) of two compactly supported probability distributions $\mu_{1,2}$: sample $X_{1,2} \in \mathbb{R}^n$ from $\mu_{1,2}$ and consider

$$A = U_1 \mathrm{diag}(X_1) U_1^* + U_2 \mathrm{diag}(X_2) U_2^*,$$

where $U_{1,2}$ are $n \times n$ independent Haar unitary rotations. Then, as $n \to \infty$, the spectrum of A has distribution $\mu_1 \boxplus \mu_2$.

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- The free Poisson distribution of parameter $c > 0$:

$$\pi_c = \max(1-c, 0)\delta_0 + \frac{\sqrt{4c - (x-1-c)^2}}{2\pi x} \mathbf{1}_{[(1-\sqrt{c})^2, (1+\sqrt{c})^2]}(x)\, dx.$$

- One has a free Poisson Central Limit Theorem:

$$\lim_{n\to\infty} \left[\left(1 - \frac{c}{n}\right) \delta_0 + \frac{c}{n}\delta_1 \right]^{\boxplus n} = \pi_c.$$

- Moreover, π_c is the limit eigenvalue distribution of a rescaled density matrix from the induced ensemble $\rho_{d,cd}$ (d large).

Threshold for $\mathcal{PPT}$, unbalanced & balanced case

Theorem (unbalanced case, Banica, N.)

In the unbalanced case $d_1 = d \to \infty$, $d_2 = n$ fixed, the lower bound of a threshold for $\mathcal{PPT}$ is given by $s_0 = \left[2 + 2\sqrt{1 - n^{-2}}\right] dn$.

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Theorem (balanced case, Aubrun - 2010)

In the balanced case $d_1 = d_2 = d \to \infty$, a threshold pair for $\mathcal{PPT}$ is given by $s_0 = s_1 = 4d^2$.

Thank you !

http://arxiv.org/abs/1105.2556

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work in progress