Random repeated quantum interactions and random invariant states

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Repeated quantum interactions and quantum channels

Invariant states and the asymptotic induced ensemble

Random environment states and i.i.d. interaction unitaries

Closed systems quantum dynamics

- Schrödinger equation

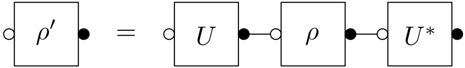
$$|\psi'\rangle = U|\psi\rangle,$$

where $U = e^{-i\tau H} \in \mathcal{U}(d)$ is the unitary interaction matrix

- For density matrices

$$\rho' = U\rho U^*$$

- Graphical form



Open systems quantum dynamics

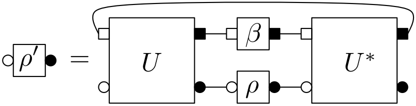
- The system $\rho \in \mathcal{M}_d^{1,+}(\mathbb{C})$ is coupled to a (possibly unknown/inaccessible) environment described by a state $\beta \in \mathcal{M}_{d'}^{1,+}(\mathbb{C})$ and undergoes a "closed" unitary dynamics described by a matrix $U \in \mathcal{U}(dd')$

$$U(\rho \otimes \beta)U^*$$

- Since we do not have access to the environment, we perform a partial trace over the d' "hidden" degrees of freedom

$$\rho' = \mathrm{Tr}_{d'}\left[U(\rho \otimes \beta)U^*\right]$$

- Graphical form



Quantum channels

A quantum channel is a *completely positive*, *trace preserving* linear map $\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$.

- Φ is completely positive if, for all $d' \geqslant 1$, the map

$$\Phi \otimes I_{d'} : \mathcal{M}_{dd'}(\mathbb{C}) \to \mathcal{M}_{dd'}(\mathbb{C})$$

  is positive;

- Φ is trace preserving if

$$\mathrm{Tr}\left[\Phi(X)\right] = \mathrm{Tr}\left[X\right], \quad \forall X \in \mathcal{M}_d(\mathbb{C}).$$

Remark

The transposition map

$$\begin{aligned} T : \mathcal{M}_d(\mathbb{C}) &\to \mathcal{M}_d(\mathbb{C}) \\ X &\mapsto X^\top \end{aligned}$$

is not completely positive.

Quantum channels

A linear map $\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ is a quantum channel iff one of the following holds:

1. Stinespring dilation There exists a finite dimensional Hilbert space $\mathbb{C}^{d'}$, a density matrix $\beta \in \mathcal{M}_{d'}^{1,+}(\mathbb{C})$ and an unitary operation $U \in \mathcal{U}(dd')$ such that

$$\Phi(X) = \mathrm{Tr}_{d'}\left[U(X \otimes \beta)U^*\right], \quad \forall X \in \mathcal{M}_d(\mathbb{C}).$$

2. Kraus decomposition There exists an integer k and matrices $L_1, \ldots, L_k \in \mathcal{M}_d(\mathbb{C})$ such that

$$\Phi(X) = \sum_{i=1}^{k} L_i X L_i^*, \quad \forall X \in \mathcal{M}_d(\mathbb{C})$$

and

$$\sum_{i=1}^{k} L_i^* L_i = I_d.$$

Quantum channels

- Stinespring dilation

$$\Phi(X) = \mathrm{Tr}_{d'}\left[U(X \otimes \beta)U^*\right];$$

- Kraus decomposition

$$\Phi(X) = \sum_{i=1}^{k} L_i X L_i^*, \quad \sum_{i=1}^{k} L_i^* L_i = I_d.$$

It can be shown that the dimension of the ancilla space in the Stinespring dilation theorem can be chosen $d' = d^2$ and β can be chosen to be a rank one projector. A similar result holds for the number of Kraus operators: one can always find a decomposition with $k = d^2$ operators.
The *Choi rank* of a quantum channel Φ is the least positive integer k such that Φ admits a Kraus decomposition with k operators L_i.

Two examples

- For $U \in \mathcal{U}(d)$, define the unitary conjugation channel

$$\Phi_U(X) = UXU^*.$$

One can check that the spectrum of Φ_U is

$$\text{spec}(\Phi_U) = \{\lambda_1 \overline{\lambda}_2 \mid \lambda_1, \lambda_2 \in \text{spec}(U)\}.$$

For $U = I$, one gets the identity channel $\Phi_I(X) = X$.

- The depolarizing channel $\Phi_{\text{dep}} : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ is given by

$$\Phi_{\text{dep}}(X) = \text{Tr}(X)\frac{I}{d}.$$

It has eigenvalues 1 (with multiplicity 1) and 0 (with multiplicity $d^2 - 1$).

Spectral properties of channels

Proposition

Let $\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ *a quantum channel. Then*

1. Φ *has at least one invariant element which is a density matrix;*
2. Φ *has trace operator norm 1;*
3. Φ *has spectral radius 1;*
4. Φ *satisfies the Schwarz inequality*

$$\forall X \in \mathcal{M}_d(\mathbb{C}), \quad \Phi(X)^* \Phi(X) \leqslant \|\Phi(I)\| \, \Phi(X^* X).$$

Asymptotic states for a class of channels

Let $\mathcal{C}$ be the set of all quantum channels that have 1 as a simple eigenvalue and all other eigenvalues are contained in the open unit disc.

Proposition

Consider a quantum channel $\Phi \in \mathcal{C}$. Then, for all density matrices $\rho_0 \in \mathcal{M}_d^{1,+}(\mathbb{C})$,

$$\lim_{n\to\infty} \Phi^n(\rho_0) = \rho_\infty,$$

where ρ_∞ is the unique invariant state of Φ.

A model of random quantum channels

Fix two integers $d, d' \geqslant 2$ and a density matrix $\beta \in \mathcal{M}_{d'}^{1,+}(\mathbb{C})$. To an unitary matrix $U \in \mathcal{U}(dd')$, associate the channel

$$\Phi^{U,\beta}(X) = \mathrm{Tr}_{d'}\left[U(X \otimes \beta)U^*\right].$$

Choosing U random from the Haar distribution $\mathfrak{h}_{dd'}$ on the unitary group, we obtain a quantum channel-valued random variable (β is fixed)

$$\begin{aligned}\mathcal{U}(dd') &\to \mathcal{L}(\mathcal{M}_d(\mathbb{C}))\\ U &\mapsto \Phi^{U,\beta}.\end{aligned}$$

Question: What are the properties of a *generic* quantum channel ?

Almost all quantum channels are in $\mathcal{C}$

Theorem

Let β be a fixed density matrix of size d'. If U is a random unitary matrix distributed along the Haar invariant probability $\mathfrak{h}_{dd'}$ on $\mathcal{U}(dd')$, then $\Phi^{U,\beta} \in \mathcal{C}$ almost surely.

Corollary

For almost all unitary matrices $U \in \mathcal{U}(dd')$, the channel $\Phi^{U,\beta}$ has an unique invariant state ρ_∞ and for all density matrices ρ_0,

$$\lim_{n\to\infty} \left(\Phi^{U,\beta}\right)^n (\rho_0) = \rho_\infty.$$

Strictly positive and irreducible channels

Definition

A positive map $\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ is called

- strictly positive (or positivity improving) if $\Phi(X) > 0$ for all $X \geqslant 0$;
- irreducible if there is no (non-trivial) projector P such that $\Phi(P) \leqslant \lambda P$ for some $\lambda > 0$.

Proposition

A positive linear map $\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ is irreducible if and only if the map $(I + \Phi)^{d-1}$ is strictly positive.

Strictly positive and irreducible channels

Theorem

If Ψ *is a unital, irreducible map on* $\mathcal{M}_d(\mathbb{C})$ *which satisfies the Schwarz inequality (eg. the dual of an irreducible quantum channel* Φ*), then the set of peripheral (i.e. modulus one) eigenvalues is a (possibly trivial) subgroup of the unit circle* $\mathbb{T}$*. Moreover, every peripheral eigenvalue is simple and the corresponding eigenspaces are spanned by unitary elements of* $\mathcal{M}_d(\mathbb{C})$*.*

Corollary

The peripheral eigenvalues of an irreducible quantum channel are simple and contained in the finite set

$$\{\xi \in \mathbb{T} \mid \exists 1 \leqslant n \leqslant d^2 \text{ s.t. } \xi^n = 1\}.$$

Necessary and sufficient conditions for irreducibility

We denote by $\mathrm{Lat}(T)$ the lattice of invariant subspaces of an operator $T \in \mathcal{M}_d(\mathbb{C})$.

Proposition (Farenick)

Consider a completely positive map $\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ *defined by*

$$\Phi(X) = \sum_{i=1}^{k} L_i X L_i^*,$$

with $L_i \in \mathcal{M}_d(\mathbb{C})$, $i = 1, \ldots, k$. *Then* Φ *is irreducible if and only if* $\bigcap_{i=1}^{k} \mathrm{Lat}(L_i)$ *is trivial.*

Necessary and sufficient conditions for irreducibility

Proposition (the Shemesh criterion)

Two matrices $A, B \in \mathcal{M}_d(\mathbb{C})$ *have a common eigenvector if and only if*

$$\bigcap_{i,j=1}^{d-1} \ker \left[A^i, B^j\right] \neq \{0\}.$$

More generally, if A and B have a common invariant subspace of dimension k (for $1 \leqslant k \leqslant d-1$*), then their k-th wedge powers have a common eigenvector, and hence (we put* $n = \binom{d}{k}$*)*

$$\bigcap_{i,j=1}^{n-1} \ker \left[(A^{\wedge k})^i, (B^{\wedge k})^j\right] \neq \{0\}.$$

Almost all quantum channels are irreducible

- Write the matrix U defining a quantum channel Φ as a $d' \times d'$ matrix of blocks in $\mathcal{M}_d(\mathbb{C})$: $U \in \mathcal{M}_{d'}(\mathcal{M}_d(\mathbb{C}))$. Then, the Kraus matrices L_i are (rescaled copies) of the blocks $U^{s,t} \in \mathcal{M}_d(\mathbb{C})$.
- The Shemesh condition on the existence of an common invariant subspace can be written as

$$\det \sum_{i,j=1}^{n-1} \left[(A^{\wedge k})^i, (B^{\wedge k})^j \right]^* \cdot \left[(A^{\wedge k})^i, (B^{\wedge k})^j \right] = 0.$$

- This is a polynomial equation in the real and imaginary parts of the $(dd')^2$ complex coefficients of the matrix U.
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Conclusion: almost all quantum channels are in $\mathcal{C}$

For almost all unitary matrices $U \in \mathcal{U}(dd')$, the channel $\Phi^{U,\beta}$ has an unique invariant state ρ_∞ (which depends on U) and for all density matrices ρ_0,

$$\lim_{n\to\infty} \left(\Phi^{U,\beta}\right)^n (\rho_0) = \rho_\infty.$$

From quantum channels to random density matrices

For a norm one vector $x \in \mathbb{C}^d$, define the random density matrix

$$\mathcal{U}(dd') \ni U \mapsto \rho = \Phi^{U,\beta}(|x\rangle\langle x|) = \mathrm{Tr}_{d'}[U(|x\rangle\langle x| \otimes \beta)U^*].$$

If $\beta = |y\rangle\langle y|$ ($y \in \mathbb{C}^{d'}$) is a rank-one projector, then

$$\rho = \mathrm{Tr}_{d'} |U(x \otimes y)\rangle\langle U(x \otimes y)|$$

is an element of the induced density matrices ensemble (of parameters d, d').

- the distribution of ρ *does not* depend on the choice of the unit vectors x and y;
- ρ has the same distribution as $\mathrm{Tr}_{d'} |z\rangle\langle z|$, where z is a Lebesgue-uniform vector on the unit sphere of $\mathbb{C}^d \otimes \mathbb{C}^{d'} \simeq \mathbb{C}^{dd'}$.

The induced ensemble

$$\rho = \mathrm{Tr}_{d'} |z\rangle\langle z|, \quad z \text{ uniform on } \mathbb{S}(\mathbb{C}^d \otimes \mathbb{C}^{d'})$$

- The distribution of ρ is unitarily invariant: $\rho \overset{\text{law}}{=} V\rho V^*$ for all $V \in \mathcal{U}(d)$. Hence ρ diagonalizes $\rho = V\,\mathrm{diag}(\delta)V^*$, where δ is a random vector in the probability simplex and V is a Haar unitary;
- There is a connection with the Wishart ensemble from Random Matrix Theory: if $W = XX^*$ is a Wishart matrix of parameters d and d', then

$$\rho \overset{\text{law}}{=} \frac{W}{\mathrm{Tr}\, W};$$

- Asymptotics, in the regimes [d fixed, $d' \to \infty$], [$d \to \infty$ fixed, d' fixed] and [$d, d' \to \infty$, $d'/d \to c > 0$] are well understood.

A new model of random density matrices

- induced ensemble = one iteration of a random channel

$$\begin{aligned}\rho &= \mathrm{Tr}_{d'}[U(|x\rangle\langle x| \otimes |y\rangle\langle y|)U^*] \\ &= \Phi^{U,y}(|x\rangle\langle x|).\end{aligned}$$

- What about a large number of iterations ? For almost all U,

$$\left(\Phi^{U,y}\right)^n (|x\rangle\langle x|) \underset{n\to\infty}{\longrightarrow} \rho_\infty.$$

- For a Haar-distributed unitary U, the distribution of ρ_∞ *does not* depend on x and y.
- We have defined an ensemble of density matrices

$$\mathcal{U}(dd') \ni U \mapsto \rho_\infty = \lim_{n\to\infty} \left(\Phi^{U,y}\right)^n (|x\rangle\langle x|).$$

The asymptotic induced ensemble

Fix $d, d' \geqslant 2$ and a probability vector $b \in \mathbb{C}^{d'}$. An element from the asymptotic induced ensemble of parameters (d, b) is the random density matrix

$$\mathcal{U}(dd') \ni U \mapsto \rho_\infty = \lim_{n\to\infty} \left(\Phi^{U,\beta}\right)^n (\rho_0),$$

where

- $\beta \in \mathcal{M}_{d'}^{1,+}(\mathbb{C})$ is a fixed density matrix with spectrum b;
- $\rho_0 \in \mathcal{M}_{d}^{1,+}(\mathbb{C})$ is a fixed initial state.

Remarks

- The map $U \mapsto \rho_\infty$ is defined almost everywhere on $\mathcal{U}(dd')$ (almost all random channels are in $\mathcal{C}$).
- The distribution of ρ_∞ does not depend on ρ_0 and on the "phase" of β; it depends only on the eigenvalue vector $b = \mathrm{spec}(\beta)$.
- The distribution of ρ_∞ is unitarily invariant: $\rho_\infty \overset{\text{law}}{=} V\rho_\infty V^*$ for all $V \in \mathcal{U}(d)$.

Numerical simulations

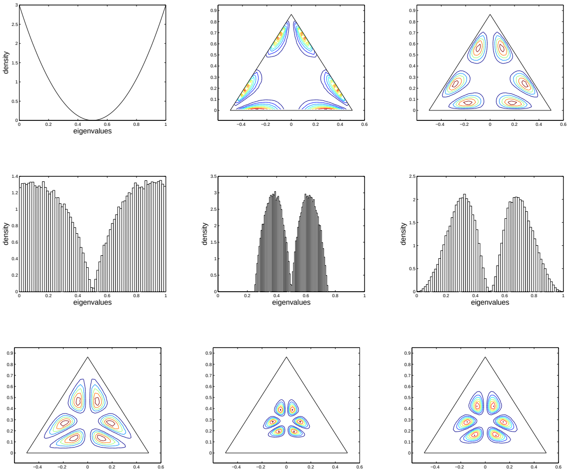

Figure: First row - induced measure d'=2, d'=3, d'=5; Second & third rows - asymptotic measure b=[1, 0], b=[3/4, 1/4],b=[1, 0, 0, 0];b=[1, 0, 0],b=[3/4, 1/8, 1/8] and b=[1, 0, 0, 0, 0].

Models of random repeated quantum interactions

Repeated quantum interactions

$$\rho_n = \Phi^{U_n,\beta_n}(\rho_{n-1}) = \mathrm{Tr}_{d'}\left[U_n(\rho_{n-1} \otimes \beta_n)U_n^*\right].$$

The first model of random repeated interactions we studied was given by the iteration of the channel

$$\begin{aligned} \mathcal{U}(dd') &\to \mathcal{L}(\mathcal{M}_d(\mathbb{C})) \\ U &\mapsto \Phi^U. \end{aligned}$$

We now introduce two new models:

- random environment: U is fixed, and the successive environment states $(\beta_n)_n$ are i.i.d. random density matrices.
- i.i.d. unitaries: the sequence of interaction unitaries $(U_n)_n$ is Haar-i.i.d., and no assumption is made on the (possibly random) environment states $(\beta_n)_n$.

Asymptotic results: random environment

Discrete evolution equation

$$\rho_n = \Phi^{\beta_n}(\rho_{n-1}) = \mathrm{Tr}_{d'}\left[U(\rho_{n-1} \otimes \beta_n)U^*\right].$$

In this model, the interaction unitary U is fixed beforehand and the environment states $(\beta_n)_n$ are i.i.d. random density matrices.
As usual, we are interested in the asymptotic behavior of the states

$$\rho_n = \Phi^{\beta_n} \circ \cdots \circ \Phi^{\beta_1}(\rho_0).$$

We use results by L. Bruneau, A. Joye and M. Merkli on products of random matrices, applied to the (i.i.d.) channels

$$\Phi^{\beta_n} \in \mathcal{L}(\mathcal{M}_d(\mathbb{C})).$$

Asymptotic results: random environment

Theorem (BJM)

Let $(M_n)_n$ be a sequence of i.i.d. random contractions of $\mathcal{M}_D(\mathbb{C})$ with the following properties:

1. *There exists a constant vector $\psi \in \mathbb{C}^D$ such that $M\psi = \psi$ almost surely;*
2. $\mathbb{P}($*1 is a simple eigenvalue of M*$) > 0$.

Then the (deterministic) matrix $\mathbb{E}[M]$ has eigenvalue 1 with multiplicity one and there exists a constant vector $\theta \in \mathbb{C}^D$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} M_1(\omega) M_2(\omega) \cdots M_n(\omega) = |\psi\rangle\langle\theta| = P_{1,\mathbb{E}[M]},$$

where $P_{1,\mathbb{E}[M]}$ is the rank-one spectral projector of $\mathbb{E}[M]$ corresponding to the eigenvalue 1.

Asymptotic results: random environment

Using the duality between the Schrödinger and the Heisenberg pictures of Quantum Mechanics, we obtain

Theorem

Let $(\Phi_n)_n$ be a sequence of i.i.d. random quantum channels acting on $\mathcal{M}_d(\mathbb{C})$ such that

$$\mathbb{P}(\Phi \textit{ has an unique invariant state}) > 0.$$

Then $\mathbb{E}[\Phi]$ is a quantum channel with an unique invariant state $\theta \in \mathcal{M}_d^{1,+}(\mathbb{C})$ and, $\mathbb{P}$-almost surely,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} [\Phi_n \circ \cdots \circ \Phi_1](\rho_0) = \theta, \quad \forall \rho_0 \in \mathcal{M}_d^{1,+}(\mathbb{C}).$$

Asymptotic results: random environment

Proposition

Let $\{\beta_n\}_n$ be a sequence of i.i.d. random density matrices such that, with positive probability, the random quantum channel Φ^{β} has an unique invariant state. Then, almost surely, for all initial states $\rho_0 \in \mathcal{M}_d^{1,+}(\mathbb{C})$, one has

$$\lim_{N\to\infty} \frac{\rho_1 + \ldots + \rho_N}{N} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} [\Phi^{\beta_n} \circ \cdots \circ \Phi^{\beta_1}](\rho_0) = \theta,$$

where $\theta \in \mathcal{M}_d^{1,+}(\mathbb{C})$ is the unique invariant state of the deterministic channel $\Phi^{\mathbb{E}[\beta]}$.
In particular, if $\mathbb{E}[\beta] = I_{d'}/d'$, then θ is the "chaotic" state I_d/d.

Asymptotic results: i.i.d. unitaries

Discrete evolution equation

$$\rho_n = \Phi^{U_n,\beta_n}(\rho_{n-1}) = \mathrm{Tr}_{d'}\left[U_n(\rho_{n-1} \otimes \beta_n)U_n^*\right].$$

In this model, the interaction unitaries U_n are Haar distributed independent random matrices.
The environment states $(\beta_n)_n$ are independent of the family $(U_n)_n$ and can have an arbitrary joint distribution.

Lemma

Let $(V_n)_n$ be a sequence of i.i.d. Haar unitaries independent of the family $\{U_n, \beta_n\}_n$ and consider the sequence of successive states $(\rho_n)_n$ defined earlier. Then the sequences $(\rho_n)_n$ and $(V_n \rho_n V_n^*)_n$ have the same distribution.

Asymptotic results: i.i.d. unitaries

Lemma

The sequence of successive states $(\rho_n)_n$ and its i.i.d.-randomly rotated version $(V_n \rho_n V_n^*)_n$ have the same distribution.

Consequences

- ρ_n fluctuates, hence the need for an ergodic theorem.
- in the ergodic sum, the "phases" are random, uniform and independent of the rest.

Proposition

Let $(\rho_n)_n$ be the successive states of a repeated quantum interaction scheme with i.i.d. random unitary interactions. Then, almost surely,

$$\lim_{n\to\infty} \frac{\rho_1 + \ldots + \rho_n}{n} = \frac{I_d}{d}.$$

Conclusion and perspectives

- Study other properties of random quantum channels, such as minimal output entropies
- Connections to Hayden's and Hastings' counterexamples to the additivity conjecture
- Statistical properties of the asymptotic induced ensemble (support of the measure, moments, mean entropy)
- Study the asymptotic induced ensemble for large matrices: limit theorems for the empirical eigenvalue distribution, convergence of the extremal eigenvalues, fluctuations
- Random scalings of random invariant states $\Rightarrow$ random positive matrices (à la Wishart)
- Continuous limit ?

Thank you !

http://arxiv.org/abs/0902.2634