Random quantum channels
- graphical calculus -

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joint work with Benot Collins

Fields Workshop on Operator Structures in Quantum Information

Toronto, July 9, 2009

Random quantum channels
&
additivity problems

Additivity for MOE of quantum channels

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$$H^p_{\min}(\Phi) = \min_{\rho \in \mathcal{M}_{\text{in}}(\mathbb{C})} H^p(\Phi(\rho)) = \min_{x \in \mathbb{C}^{\text{in}}} H^p(\Phi(P_x)).$$

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- NO !!!
  - $p > 1$: Hayden '07, Hayden + Winter '08
  - $p = 1$: Hastings '09

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- Equivalently, via the Stinespring dilation theorem

$$\Phi(\rho) = \mathrm{Tr}_{\mathrm{aux}}(U(\rho \otimes P_y) U^*),$$

where $y \in \mathbb{C}^{\frac{\mathrm{out} \times \mathrm{aux}}{\mathrm{in}}}$ and $U \in \mathcal{M}_{\mathrm{out} \times \mathrm{aux}}(\mathbb{C})$ is a Haar unitary matrix.

Our model

Choice of parameters

- in $= tnk$,
- out $= k$,
- aux $= n$,

where $n, k \in \mathbb{N}$ and $t \in (0, 1)$. In general, we shall assume that

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We are thus considering random channels

$$\begin{aligned} \Phi : \mathcal{M}_{tnk}(\mathbb{C}) &\to \mathcal{M}_k(\mathbb{C}) \\ \rho &\mapsto \mathrm{Tr}_n \left[U(\rho \otimes P_y) U^* \right], \end{aligned}$$

where $y \in \mathbb{C}^{t^{-1}}$ is fixed (and irrelevant) and $U \in \mathcal{U}(nk)$ is a Haar random unitary matrix.

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Strategy

- Use trivial bound

$$H^p_{\min}(\Phi \otimes \overline{\Phi}) \leqslant H^p\left([\Phi \otimes \overline{\Phi}](X_{12})\right),$$

for a particular choice of $X_{12} \in \mathcal{M}_{tnk}(\mathbb{C}) \otimes \mathcal{M}_{tnk}(\mathbb{C})$.

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- Bound entropies of the (random) density matrix

$$Z = [\Phi \otimes \overline{\Phi}](E_{tnk}) \in \mathcal{M}_{k^2}(\mathbb{C}).$$

Main result for product channel

Theorem (Collins + N. '09)

For all k, t, almost surely as $n \to \infty$, the eigenvalues of $Z = [\Phi \otimes \overline{\Phi}](E_{tnk})$ converge to

$$\left(t + \frac{1-t}{k^2}, \underbrace{\frac{1-t}{k^2}, \ldots, \frac{1-t}{k^2}}_{k^2-1 \text{ times}} \right).$$

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- However, smaller eigenvalues are the "worst possible".
- Precise knowledge of eigenvalue $\rightsquigarrow$ optimal estimates for entropies.

Graphical calculus for random quantum channels

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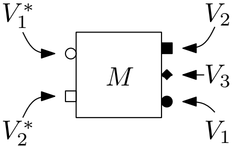

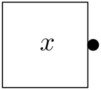

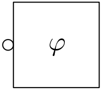

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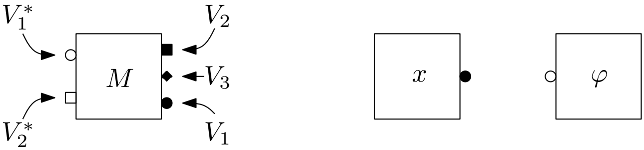

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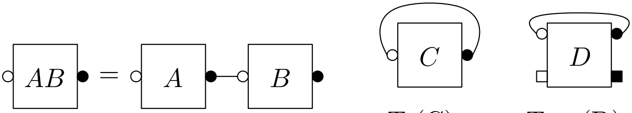
- Tensor contractions (or traces) $V \otimes V^* \to \mathbb{C} \rightsquigarrow$ wires.



$\mathrm{Tr}(\mathrm{C})$ $\qquad \mathrm{Tr}_{\mathrm{V}_1}(\mathrm{D})$

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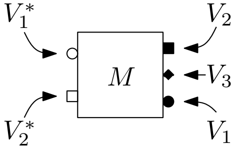

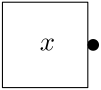

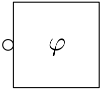

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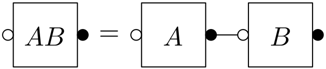
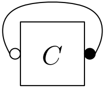
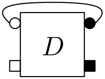


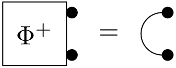


Tr(C)



$\mathrm{Tr}_{V_1}(D)$

- Bell state $\Phi^+ = \sum_{i=1}^{\dim V_1} e_i \otimes e_i \in V_1 \otimes V_1$



Graphical representation of quantum channels

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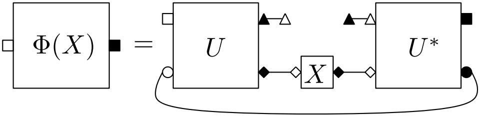
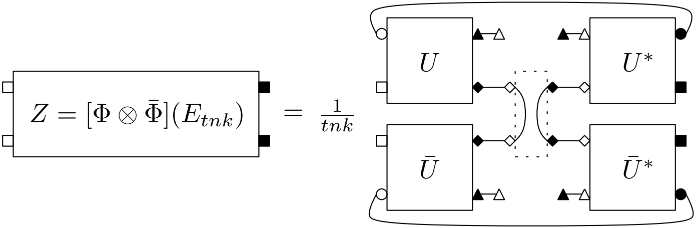


- Product of conjugate channels



- Decorations/labels

●/○ $= \mathbf{C}^n$ ■/□ $= \mathbf{C}^k$ ◆/◇ $= \mathbf{C}^{tnk}$ ▲/△ $= \mathbf{C}^{t^{-1}}$

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- Example

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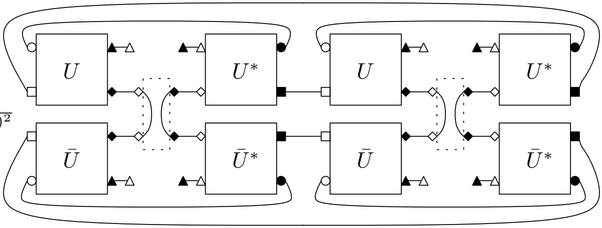

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Theorem (Weingarten formula)

Let d be a positive integer and $\mathbf{i} = (i_1, \ldots, i_p)$, $\mathbf{i}' = (i'_1, \ldots, i'_p)$, $\mathbf{j} = (j_1, \ldots, j_p)$, $\mathbf{j}' = (j'_1, \ldots, j'_p)$ be p-tuples of positive integers from $\{1, 2, \ldots, d\}$. Then

$$\int_{\mathcal{U}(d)} U_{i_1 j_1} \cdots U_{i_p j_p} \overline{U_{i'_1 j'_1}} \cdots \overline{U_{i'_p j'_p}} \, dU = \sum_{\alpha, \beta \in \mathcal{S}_p} \delta_{i_1 i'_{\alpha(1)}} \ldots \delta_{i_p i'_{\alpha(p)}} \delta_{j_1 j'_{\beta(1)}} \ldots \delta_{j_p j'_{\beta(p)}} \operatorname{Wg}(d, \alpha\beta^{-1}).$$

If $p \neq p'$ then

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- There is a graphical way of reading this formula on the diagrams !

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4. For all $i = 1, \ldots, p$, add a wire between each white decoration of the i-th U box and the corresponding white decoration of the $\alpha(i)$-th $\overline{U}$ box. In a similar manner, use β to pair black decorations.

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5. Erase all U and $\overline{U}$ boxes. The resulting diagram is denoted by $\mathcal{D}_{(\alpha,\beta)}$.

Theorem

$$\mathbb{E}\mathcal{D} = \sum_{\alpha,\beta} \mathcal{D}_{(\alpha,\beta)} \operatorname{Wg}(d, \alpha\beta^{-1}).$$

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- After doing the loop combinatorics, one is left with maximizing over S_{2p}^2 quantities such as

$$\#(\gamma^{-1}\alpha) + \#(\alpha^{-1}\beta) + \#(\beta^{-1}\delta),$$

where γ and δ are permutations coding the initial wiring of $U/\overline{U}$ boxes and $\#(\cdot)$ is the number of cycles function.

Application: product conjugate channel

- We want to compute, for all $p \geqslant 1$, $\mathbb{E}\,\mathrm{Tr}(Z^p)$.
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- Geodesic problems in symmetric groups $\Rightarrow$ non-crossing partitions $\Rightarrow$ free probability.
- Asymptotic for Weingarten weights:

$$\mathrm{Wg}(d,\sigma) = d^{-(p+|\sigma|)}(\mathrm{Mob}(\sigma) + O(d^{-2})).$$

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- Improved bounds for MOE of product channels
- Other applications to QIT (work in progress with B. Collins and K. Życzkowski)

Thank you !

Next talk $\rightsquigarrow$ bounds for 1 channel

http://arxiv.org/abs/0905.2313
http://arxiv.org/abs/0906.1877