A mathematical introduction to Quantum Information Theory

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Outline

1. Quantum information theory
2. Pure states
3. Density matrices
4. Quantum channels

"The big secret of quantum mechanics is how simple it is once you take the physics out of it."

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Quantum Information Theory

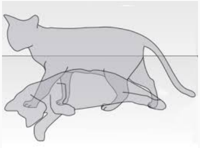
The theory of quantum information is composed of two subfields

1. quantum computing: quantum algorithms
2. quantum Shannon theory: protocols for (secure) transmission of (quantum) data

In order to achieve better performance/speed than the classical theory, quantum information harnesses purely quantum phenomena such as:

- Quantum superposition

The state space of a quantum system is a vector space. In the classical theory, information is stored into bits, which can only take the discrete set of values 0 et 1. A qubit is a unit norm vector of $\mathbb{C}^2 = \text{span}\{|0\rangle, |1\rangle\}$.



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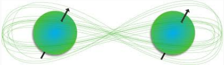
- Entanglement

There exist quantum states of a multi-partite system which can not be described only in terms of the individual subsystems.



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A brief history of quantum information theory

- 1982: Feynman suggests using a quantum computer to efficiently simulate quantum systems
- 1984: Bennett and Brassard invent a protocol (BB84) using quantum mechanics to securely distribute cryptographic keys
- 1989: BB84 demonstrated experimentally
- 1992: Deutsch and Jozsa formulate the first quantum algorithm outperforming the best possible classical algorithm for the same task
- 1994: Shor discovers a quantum factoring algorithm: N can be factored on a quantum computer in $O(\log^3 N)$ vs. $O(\exp(\log^{1/3} N))$ for the best known classical algorithm
- 2012: $21 = 3 \times 7$ factored on a quantum computer using photons
- 2015: D-Wave Systems, the first quantum computing company, announces a (non-universal) quantum computer using 1000 qubits
- 2018: The race towards quantum supremacy: approx. 50-100 qubits

Quantum states. Entanglement

Quantum states - the big picture

- One quantum system

States	Deterministic	Random mixture
Classical	$x \in \{1, 2, \dots, d\}$	$p \in \mathbb{R}^d,\ p_i \geq 0,\ \sum_i p_i = 1$
Quantum	$\psi \in \mathbb{C}^d,\ \|\psi\| = 1$	$\rho \in \mathcal{M}_d(\mathbb{C}),\ \rho \geq 0,\ \operatorname{Tr} \rho = 1$

- Two (or more) quantum systems: tensor product of individual systems (at the level of Hilbert spaces or at the level of matrices)

$$\Downarrow$$

entanglement

Axioms of Quantum Mechanics with pure states

- To every quantum mechanical system, we associate a Hilbert space $\mathcal{H} \cong \mathbb{C}^d$. The state of a system is described by a unit vector $|\psi\rangle \in \mathcal{H}$.

Example

The qubit - a two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$. States in superposition are allowed: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $\{|0\rangle, |1\rangle\}$ is an orthonormal basis of $\mathbb{C}^2$; we have $|\alpha|^2 + |\beta|^2 = 1$.

- States evolve according to unitary transformations $U \in \mathcal{U}(d)$: $|\psi\rangle \mapsto U|\psi\rangle$. Physically, $U = \exp(-itH)$ for an Hamiltonian H.
- Observable quantities correspond to Hermitian operators $A \in \mathcal{B}(\mathcal{H})$. Let $A = \sum_i \lambda_i P_i$ be the spectral decomposition of A. Born's rule asserts that, when measuring a quantum system in state $|\psi\rangle$,

$$\mathbb{P}[\text{ we observe } \lambda_i\,] = \langle\psi|P_i|\psi\rangle$$

and that, conditionally on observing λ_i, the system's state collapses to

$$|\psi'\rangle = \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}}.$$

A basic uncertainty relation

Consider the three Pauli observables

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Proposition

For any qubit state $|\psi\rangle \in \mathbb{C}^2$ we have

$$|\langle\psi|X|\psi\rangle| + |\langle\psi|Y|\psi\rangle| + |\langle\psi|Z|\psi\rangle| \leq \sqrt{3}$$

The proof follows from the anti-commutation property of the Pauli matrices, which implies $\|X \pm Y \pm Z\| \leq \sqrt{3}$. The uncertainty in the measurement of (say) X in the state $|\psi\rangle$ is given by

$$u_X = 1 - \max\{\mathbb{P}(1), \mathbb{P}(-1)\} = \frac{1 - |\langle\psi|X|\psi\rangle|}{2}$$

By the result, the total uncertainty is lower bounded by

$$u = u_X + u_Y + u_Z \geq \frac{3 - \sqrt{3}}{2} \approx 0.63$$

Composite systems. Entanglement

For a system composed of two parts A (Alice, ) and B (Bob, ), with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the total Hilbert space is the tensor product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

A general two-qubit state $|\psi\rangle_{AB} \in \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$ is given by

$$|\psi\rangle_{AB} = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle,$$

where $|ij\rangle = |i\rangle \otimes |j\rangle$, and α_{ij} are complex amplitudes.

Definition

A pure state $|\psi\rangle_{AB}$ is called separable if $|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$.
Non-separable states are called entangled.

Entangled states are a key resource in quantum information, needed to obtain the computational speedups or to guarantee security of cryptographic protocols.

Separable states: $|\psi\rangle_{AB} = |00\rangle$ or $|\varphi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$
Entangled state: the Bell state $|\Omega\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Pure state entanglement is generic

Bipartite states can be seen as (rectangular matrices), via the isomorphism $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \cong \mathcal{M}_{d_A \times d_B}(\mathbb{C})$.

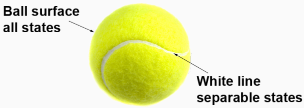
Proposition — Schmidt decomposition

Given any quantum state $|\psi\rangle_{AB}$ there exist orthonormal families $\{|e_i\rangle\}_{i=1}^r \subseteq \mathbb{C}^{d_A}$, $\{|f_i\rangle\}_{i=1}^r \subseteq \mathbb{C}^{d_B}$ and a probability vector p such that

$$|\psi\rangle = \sum_{i=1}^r \sqrt{p_i} |e_i\rangle \otimes |f_i\rangle.$$

A state is pure iff $p = (1, 0, \ldots, 0)$ iff the corresp. matrix is rank one. The Shannon entropy of p is called the entanglement entropy of $|\psi\rangle$.

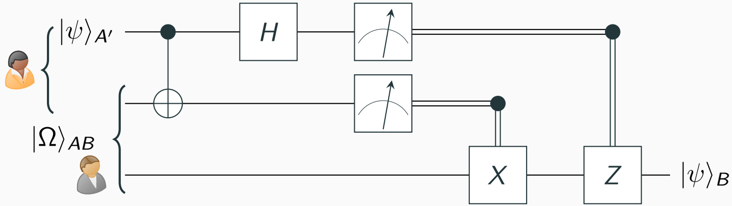
All bi-partite quantum pure states have dimension $d_A d_B - 1$, whereas product states have dimension $d_A + d_B - 2$, which is strictly smaller $\implies$ a generic pure state is entangled!



Quantum teleportation

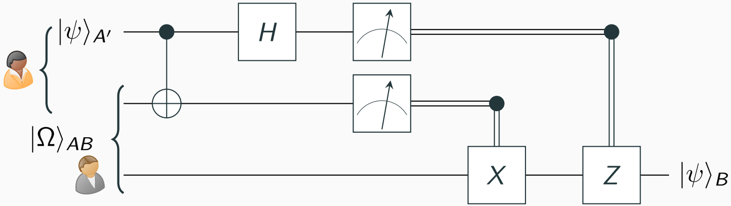
One of the first quantum protocols, discovered by Bennett, Brassard, Crépeau, Jozsa, Peres, and Wootters in 1993.

Alice wants to transmit to Bob an unknown quantum state $|\psi\rangle \in \mathbb{C}^2$. They only have access to classical communication and to a shared Bell state $|\Omega\rangle_{AB} = (|00\rangle + |11\rangle)/\sqrt{2} \in \mathbb{C}^4$.



$\boxed{H}$ is the Hadamard gate $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}/\sqrt{2}$. $\boxed{X}$, $\boxed{Z}$ are Pauli matrices. The double line on top signifies that they are controlled by a classical bit: the actual gate applied is G^b, where b is the control bit. $\oplus$ is the NOT gate, here controlled by a quantum bit: $CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Quantum teleportation — the protocol

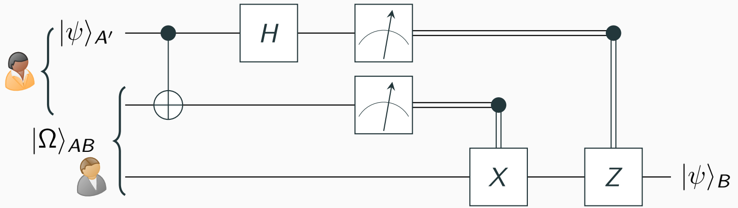


1. The system starts in the state $|\psi\rangle_{A'} \otimes |\Omega\rangle_{AB}$
2. Alice performs a CNOT operation on her 2 qubits, followed by a Hadamard gate on her A' qubit.
3. Alice measures her two qubits in the computational basis $\{|0\rangle, |1\rangle\}$.
4. Alice transmits the classical outcomes of her measurements to Bob.
5. Bob performs a controlled σ_X, followed by a controlled σ_Z gate on his qubit.

Theorem

At the end of the teleportation protocol, with probability 1, Bob's qubit is in the state $|\psi\rangle$.

Quantum teleportation — proof



$$\overbrace{(\alpha|0\rangle + \beta|1\rangle)}^{|\psi\rangle}{}_{A'} \otimes \overbrace{(|00\rangle + |11\rangle)}^{\sim|\Omega\rangle}{}_{AB} = \alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle$$

$$\xrightarrow{\mathrm{CNOT}_{A'A}} \alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle$$

$$\xrightarrow{H_{A'}} \alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle$$

$$\xrightarrow{\text{measure } A', \text{ outcome } 0} \alpha|000\rangle + \alpha|011\rangle + \beta|010\rangle + \beta|001\rangle$$

$$\xrightarrow{\text{measure } A, \text{ outcome } 1} \alpha|011\rangle + \beta|010\rangle = |01\rangle_{A'A}(\alpha|1\rangle + \beta|0\rangle)_B$$

$$\xrightarrow{X_B^1, \text{ then } Z_B^0} |01\rangle_{A'A}(\alpha|0\rangle + \beta|1\rangle)_B = |01\rangle_{A'A}|\psi\rangle_B \qquad \square$$

Mixed quantum states, a.k.a. density matrices

The Church of the larger Hilbert space

Consider a bipartite scenario $|\psi\rangle_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}d_B$, where A is the system of interest (say, an experiment) and B some "other stuff" (say, the rest of the universe).

If the system is in a product state $|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$, then measuring A with an observable X yields an expectation value

$$\langle\psi_{AB}|X \otimes I_B|\psi_{AB}\rangle = \langle\psi_A|X|\psi_A\rangle$$

In the general case where $|\psi\rangle_{AB}$ is entangled, we can write

$$\langle\psi_{AB}|X \otimes I_B|\psi_{AB}\rangle = \mathrm{Tr}\left[(X \otimes I_B)|\psi\rangle\langle\psi|_{AB}\right] = \mathrm{Tr}\left[X\rho_A\right]$$

where ρ_A is called the reduced density matrix of the state $|\psi\rangle\langle\psi|$ and it is defined by the partial trace operation

$$\rho_A = [\mathrm{id}_A \otimes \mathrm{Tr}_B](|\psi\rangle\langle\psi|) = \mathrm{Tr}_B\,|\psi\rangle\langle\psi|$$

We have thus written the expected value of measuring an observable on A as a function of an object which acts only on system A.

The partial trace

Formally, the partial trace operation is defined by linearly extending

$$\operatorname{Tr}_B(X \otimes Y) = X \cdot \operatorname{Tr}(Y) \quad \text{and} \quad \operatorname{Tr}_A(X \otimes Y) = Y \cdot \operatorname{Tr}(X)$$

In matrix notation, if

$$Z = \begin{bmatrix} Z_{11} & \cdots & Z_{1d} \\ \vdots & \ddots & \vdots \\ Z_{d1} & \cdots & Z_{dd} \end{bmatrix}, \text{ then } \operatorname{Tr}_B Z = \begin{bmatrix} \operatorname{Tr} Z_{11} & \cdots & \operatorname{Tr} Z_{1d} \\ \vdots & \ddots & \vdots \\ \operatorname{Tr} Z_{d1} & \cdots & \operatorname{Tr} Z_{dd} \end{bmatrix}$$

and $\operatorname{Tr}_A Z = Z_{11} + Z_{22} + \cdots + Z_{dd}$.

We write $\mathcal{M}^{1,+}(\mathbb{C}^d)$ for the set of density matrices

$$\mathcal{M}_d^{1,+} = \mathcal{M}^{1,+}(\mathbb{C}^d) = \{\rho \in \mathcal{M}_d(\mathbb{C}) \,:\, \rho \geq 0 \text{ and } \operatorname{Tr} \rho = 1\}$$

For pure state $|\psi\rangle = \sum_{i=1}^r \sqrt{p_i}|e_i\rangle \otimes |f_i\rangle$, we have

$$\operatorname{Tr}_B |\psi\rangle\langle\psi| = \sum_{i=1}^r \sqrt{p_i}|e_i\rangle\langle e_i| \text{ and } \operatorname{Tr}_A |\psi\rangle\langle\psi| = \sum_{i=1}^r \sqrt{p_i}|f_i\rangle\langle f_i|$$

In particular, the two partial traces have the same spectrum.

Entropy for density matrices

Recall that the Shannon entropy of a probability distribution p is $S(p) = -\sum_i p_i \log p_i$, where the log is considered in base 2, such that $S(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1) = 1$ bit.

Using functional calculus, one extends the entropy to quantum states: the von Neumann entropy

$$H(\rho) = -\operatorname{Tr}(\rho \log \rho)$$

The entanglement entropy of a bipartite quantum state is the von Neumann entropy of (any of its) reduced partial states: $E(|\psi\rangle) = H(\operatorname{Tr}_{1 \text{ or } 2} |\psi\rangle\langle\psi|) = S(p)$. A pure state $|\psi\rangle$ is separable iff $E(|\psi\rangle) = 0$ iff both its reduced density matrices are pure.

Entropy inequalities

- Bounds: $0 \leq H(\rho) \leq \log d$
- Additivity $H(\rho_A \otimes \rho_B) = H(\rho_A) + H(\rho_B)$
- Sub-additivity: $H(\rho_{AB}) \leq H(\rho_A) + H(\rho_B)$
- Strong sub-additivity: $H(\rho_{ABC}) + H(\rho_B) \leq H(\rho_{AB}) + H(\rho_{BC})$

Entanglement for density matrices

Two quantum systems: $\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$. A mixed state ρ_{AB} is called separable if it can be written as a convex combination of product states

$$\rho_{AB} \in \mathcal{SEP} \iff \rho_{AB} = \sum_i t_i \sigma_i^{(A)} \otimes \sigma_i^{(B)},$$

with $t_i \geq 0$, $\sum_i t_i = 1$, $\sigma_i^{(A,B)} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_{A,B}})$. Non-separable states are called entangled.

A pure bipartite state $\rho_{AB} = |\psi\rangle\langle\psi|$ is separable iff $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$:

$$|\psi\rangle\langle\psi| = |\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B|$$

The largest Euclidean ball centered in the maximally mixed state $I_{d_A d_B}/(d_A d_B)$ that can be inscribed in $\mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ is separable. In particular, $\mathcal{SEP}$ has positive volume. However,

$$\lim_{d\to\infty} \mathbb{P}[\rho \in \mathcal{M}^{1,+}(\mathbb{C}^d \otimes \mathbb{C}^d) \text{ is separable}] = 0.$$

Mixed state entanglement is hard, but...

Deciding if a given ρ_{AB} is separable is NP-hard. Detecting entanglement for general states is a difficult, central problem in QIT.

A linear map $f : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^{d'})$ is called

- positive if $A \geq 0 \implies f(A) \geq 0$;
- completely positive if $\mathrm{id}_k \otimes f$ is positive for all $k \geq 1$.

If $f : \mathcal{M}(\mathbb{C}^{d_B}) \to \mathcal{M}(\mathbb{C}^{d_B})$ is CP, then for every state ρ_{AB} one has $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geq 0$.

If $f : \mathcal{M}(\mathbb{C}^{d_B}) \to \mathcal{M}(\mathbb{C}^{d_B})$ is only positive, then for every separable state ρ_{AB}, one has $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geq 0$. Indeed,

$$[\mathrm{id}_{d_A} \otimes f] \left(\sum_i t_i \sigma_i^{(A)} \otimes \sigma_i^{(B)} \right) = \sum_i t_i \sigma_i^{(A)} \otimes f(\sigma_i^{(B)}) \geq 0,$$

since each term is positive semidefinite.

Entanglement detection via positive, but not CP maps

Positive, but not CP maps f yield entanglement criteria: given ρ_{AB}, if $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \not\geq 0$, then ρ_{AB} is entangled.

The following converse holds: if, for all positive maps f, $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geq 0$, then ρ_{AB} is separable.

The transposition map $\Theta(X) = X^\top$ is positive, but not CP. Let

$$\mathcal{PPT} := \{\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \,|\, [\mathrm{id}_{d_A} \otimes \Theta_{d_B}](\rho_{AB}) \geq 0\}.$$

We have $\mathcal{SEP} \subseteq \mathcal{PPT}$, with equality iff

$$(d_A, d_B) \in \{(2,2), (2,3), (3,2)\}.$$

This is the consequence of a deep result in operator algebra: every positive map $f : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_{2,3}(\mathbb{C})$ can be written as

$$f = g_1 + \Theta \circ g_2, \qquad \text{with } g_{1,2} \text{ CP}.$$

Volume-wise, for large $d_{A,B}$, $\mathcal{SEP}$ is much smaller than $\mathcal{PPT}$.

The PPT criterion at work

- Consider the Bell (or maximally entangled) state $\rho_{AB} = |\psi\rangle\langle\psi|$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B).$$

- Written as a matrix in $\mathcal{M}^{1,+}_{2\cdot 2}(\mathbb{C})$

$$\rho_{AB} = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) = \frac{1}{2}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

- Partial transposition: transpose each block B_{ij}:

$$[\mathrm{id}_2 \otimes \Theta](\rho_{AB}) = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

- This matrix is no longer positive $\implies$ the state is entangled.

Quantum channels

Quantum channels

Channels	Deterministic	Random mixture
Classical	$f : \{1, \ldots, d\} \to \{1, \ldots, d\}$	Q Markov (stochastic)
Quantum	$U \in \mathcal{U}(d)$	Φ CPTP map

- Quantum channels: CPTP maps $\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_{d'}(\mathbb{C})$
  - CP - complete positivity: $\Phi \otimes \mathrm{id}_r$ is a positive map, $\forall r \geq 1$
  - TP - trace preservation: $\mathrm{Tr} \circ \Phi = \mathrm{Tr}$.
- Example 1: unitary conjugation $\Phi(X) = UXU^*$ for a unitary matrix $U \in \mathcal{U}(d)$.
- Example 2: depolarizing channel $\Delta(X) = (\mathrm{Tr}\, X)\frac{I}{d}$.

Structure of CPTP maps

Theorem [Stinespring-Kraus-Choi]

Let $\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ be a linear map. TFAE:

1. The map Φ is completely positive and trace preserving.
2. There exist an integer n ($n = d^2$ suffices) and an isometry $V : \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^n$ such that
$$\Phi(X) = [\mathrm{id}_d \otimes \mathrm{Tr}_n](VXV^*).$$
3. There exist operators $A_1, \ldots, A_n \in \mathcal{M}_d(\mathbb{C})$ satisfying $\sum_i A_i^* A_i = I_d$ such that
$$\Phi(X) = \sum_{i=1}^{n} A_i X A_i^*.$$
4. The Choi matrix C_Φ is positive semidefinite, where
$$C_\Phi := \sum_{i,j=1}^{d} E_{ij} \otimes \Phi(E_{ij}) \in \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$$
and $[\mathrm{id} \otimes \mathrm{Tr}](C_\Phi) = I_d$.

The take-home slide

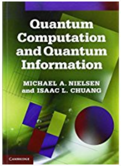
States	Deterministic	Random mixture
Classical	$x \in \{1, 2, \ldots, d\}$	$p \in \mathbb{R}^d$, $p_i \geq 0$, $\sum_i p_i = 1$
Quantum	$\psi \in \mathbb{C}^d$, $\|\psi\| = 1$	$\rho \in \mathcal{M}_d(\mathbb{C})$, $\rho \geq 0$, $\operatorname{Tr} \rho = 1$

- Quantum systems with d degrees of freedom are described by density matrices $\mathcal{M}_d^{1,+}(\mathbb{C}) = \{\rho : \operatorname{Tr} \rho = 1 \text{ and } \rho \geq 0\}$.
- Pure states are the particular case of rank one projectors, and correspond to unit vectors $\psi \in \mathbb{C}^d$; $|\psi\rangle\langle\psi| \in \mathcal{M}_d^{1,+}(\mathbb{C})$.

Channels	Deterministic	Random mixture
Classical	$f \in \mathcal{S}_d$	Q Markov: $Q_{ij} \geq 0$ and $\forall i$, $\sum_j Q_{ij} = 1$
Quantum	$U \in \mathcal{U}(d)$	Φ CPTP map

- Quantum channels: linear maps $\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_{d'}(\mathbb{C})$ which are completely positive ($\Phi \otimes \mathrm{id}_r$ is a positive map, $\forall r \geq 1$) and trace preserving ($\operatorname{Tr} \circ \Phi = \operatorname{Tr}$).
- Kraus decomposition: $\Phi(\rho) = \sum_{i=1}^k A_i \rho A_i^*$.
- Stinesrping dilation: $\Phi(\rho) = [\mathrm{id} \otimes \operatorname{Tr}](V \rho V^*)$ for an isometry V.

To go further...



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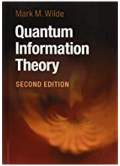

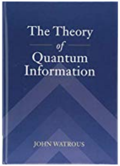

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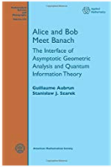

Merci!