# Quantum de Finetti theorems and Reznick's Positivstellensatz

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# Talk outline

(Quantum) de Finetti theorems

Sums of squares and Reznick's Positivstellensatz

The proof: inverting the Chiribella formula

# (Quantum) de Finetti theorems

# The classical de Finetti theorem

- Let $V$ be a finite alphabet, $|V| = d$. A probability $\mathbb{P}$ on $V^n$ is called excheangeable if it is symmetric under permutations:

$$\forall \sigma \in \mathcal{S}_n, \qquad \mathbb{P}[x_1, x_2, \ldots, x_n] = \mathbb{P}[x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}].$$

- In particular, i.i.d. distributions are exchangeable

$$\mathbb{P} = \pi^{\otimes n} \quad \text{i.e.} \quad \mathbb{P}[x_1, x_2, \ldots, x_n] = \prod_{i=1}^{n} \pi(x_i) = \prod_{a \in V} \pi(a)^{|x^{-1}(a)|}.$$

**Theorem**

Let $\mathbb{P}$ be an exchangeable probability distribution on $V^n$. Then, for $k \ll n$, its $k$-marginal $\mathbb{P}_k$ is close to a convex mixture of i.i.d. distributions. More precisely, for any $k \leq n$, there exists a probability measure $\mu$ on $\mathcal{P}(V)$ such that

$$\left\| \mathbb{P}_k - \int \pi^{\otimes k} \mathrm{d}\mu(\pi) \right\|_{\mathrm{TV}} \leq \frac{2dk}{n}.$$

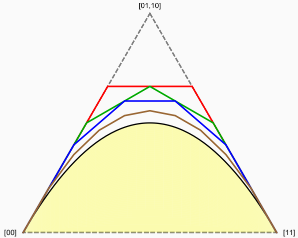



**Figure 1:** $k = 2$; $n = 3, 4, 5, 10$.

# Quantum de Finetti theorems - the setup

- Finite alphabet $[d] \rightsquigarrow$ vector space $\mathbb{C}^d$
- Probability distribution on $[d] \rightsquigarrow$ quantum state (density matrix) $\rho \in \mathcal{M}_d(\mathbb{C})$, $\rho \geq 0$, $\operatorname{Tr} \rho = 1$
- i.i.d. probability distribution $\pi^{\otimes n}$ on $[d]^{\times n} \rightsquigarrow$ multipartite product quantum state $\rho^{\otimes n} \in \mathcal{M}_d(\mathbb{C})^{\otimes n}$
- Exchangeable distribution $\mathbb{P}[x_1, \ldots, x_n] = \mathbb{P}[x_{\sigma(1)}, \ldots, x_{\sigma(n)}] \rightsquigarrow$ two different notions of symmetry for quantum states:
  1. Permutation symmetry: $\pi \rho_n \pi^* = \rho_n$, for all $\pi \in \mathcal{S}_n$
  2. Bose symmetry: $\rho_n$ supported on $\vee^n \mathbb{C}^d$, i.e. $P_{sym}^{(d,n)} \rho_n P_{sym}^{(d,n)} = \rho_n$
- Any permutationally symmetric state can be purified to a Bose symmetric pure state in $\vee^n(\mathbb{C}^d \otimes \mathbb{C}^d)$

# The finite quantum de Finetti theorem

**Theorem.**

Let $\rho \in \mathcal{B}(\vee^n \mathbb{C}^d)$ be a (Bose symmetric) quantum state. Then, for all $k \leq n$, there exists a probability measure $\mu_\rho$ on the unit sphere of $\mathbb{C}^d$ such that

$$\| \operatorname{Tr}_{n \to k} \rho - \int |\varphi\rangle\langle\varphi|^{\otimes k} \mathrm{d}\mu_\rho(\varphi) \|_1 \leq \frac{2k(d+k)}{n+d}$$

A better upper bound of $2dk/n$ can be obtained by similar methods.

- Application: the DPS hierarhy. The convex body of separable states

$$\mathrm{SEP} = \mathrm{conv}\{|x\rangle\langle x| \otimes |y\rangle\langle y| \,:\, x \in \mathbb{C}^{d_A},\, y \in \mathbb{C}^{d_B}\}$$

is hard to approximate

- A quantum state $\rho_{AB}$ is said to be *k*-extendible if $\exists\, \sigma_{AB_1 \cdots B_k}$ such that $\sigma_{B_1 \cdots B_k} \in \mathcal{B}(\vee^k \mathbb{C}^{d_B})$ and $\sigma_{AB_1} = \rho_{AB}$

**Theorem**

A state $\rho_{AB}$ is separable iff it is $k$-extendible for all $k \geq 1$

# The measure-and-prepare map

- Let $d[n] := \dim P_{sym}^{(d,n)} = \binom{n+d-1}{d-1}$ the dimension of the symmetric subspace
- Define $\mathrm{MP}_{n\to k} : \mathcal{B}(\vee^n \mathbb{C}^d) \to \mathcal{B}(\vee^k \mathbb{C}^d)$ by

$$\mathrm{MP}_{n\to k}(X) = d[n] \int \langle \varphi^{\otimes n} | X | \varphi^{\otimes n} \rangle |\varphi\rangle\langle\varphi|^{\otimes k} \mathrm{d}\varphi,$$

  where $\mathrm{d}\varphi$ is the Lebesgue measure on the unit sphere of $\mathbb{C}^d$, or even a $n+k$ spherical design
- The linear map $\mathrm{MP}_{n\to k}$ is completely positive, and it is normalized to be trace preserving (i.e. it is a quantum channel):

$$\int |\varphi\rangle\langle\varphi|^{\otimes n} \mathrm{d}\varphi = \frac{P_{sym}^{(d,n)}}{d[n]}$$

# Chiribella's formula

- Assuming $k \leq n$, let $\mathrm{Tr}_{n\to k} : \mathcal{B}(\vee^n \mathbb{C}^d) \to \mathcal{B}(\vee^k \mathbb{C}^d)$ be the partial trace map and $\mathrm{Tr}^*_{k\to n} : \mathcal{B}(\vee^k \mathbb{C}^d) \to \mathcal{B}(\vee^n \mathbb{C}^d)$ be its dual w.r.t. the Hilbert-Schmidt scalar product

$$\mathrm{Tr}^*_{k\to n}(X) = P^{(d,n)}_{sym} \left[ X \otimes I_d^{\otimes(n-k)} \right] P^{(d,n)}_{sym}$$

- $\mathrm{Clone}_{k\to n} := \frac{d[k]}{d[n]} \mathrm{Tr}^*_{k\to n}$ is the optimal Keyl-Werner cloning quantum channel

**Theorem**

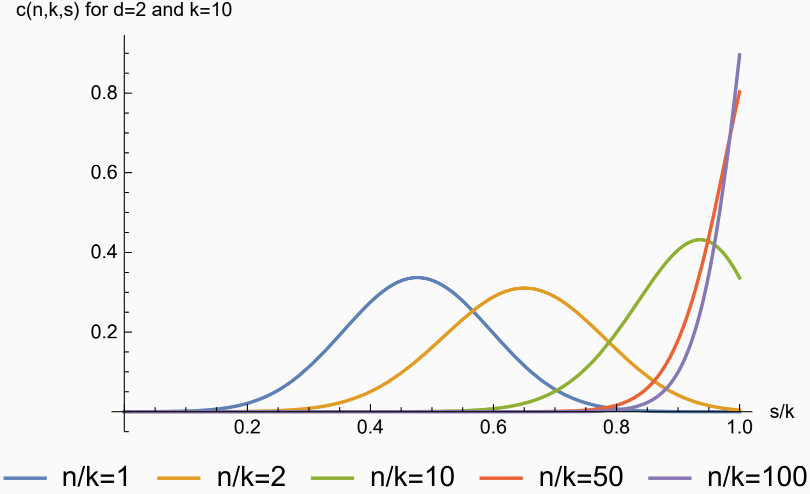
For any $k \leq n$, we have

$$\mathrm{MP}_{n\to k} = \sum_{s=0}^{k} c(n,k,s)\, \mathrm{Clone}_{s\to k} \circ \mathrm{Tr}_{n\to s},$$

where $c(n,k,s) = \binom{n}{s}\binom{k+d-1}{k-s} / \binom{n+k+d-1}{k}$.

Fact: $c(n,k,\cdot)$ is a probability distribution, $\sum_{s=0}^{k} c(n,k,s) = 1$

# Proof of the quantum de Finetti theorem



# Proof of the quantum de Finetti theorem

- Let $\|\cdot\|_\diamond$ be the $\mathcal{S}_1 \to \mathcal{S}_1$ CB norm, aka the diamond norm

$$\|\Phi\|_\diamond = \sup_k \sup_{\|X\|_1 \le 1} \|[\mathrm{id}_k \otimes \Phi](X)\|_1$$

- We have

$$\begin{aligned}
&\| \mathrm{Tr}_{n\to k} - \mathrm{MP}_{n\to k} \|_\diamond \\
&\qquad = \|(1 - c(n,k,k))\, \mathrm{Tr}_{n\to k} - \sum_{s=0}^{k-1} c(n,k,s)\, \mathrm{Clone}_{s\to k} \circ \mathrm{Tr}_{n\to s} \|_\diamond \\
&\qquad \le 2(1 - c(n,k,k)) \\
&\qquad \le \frac{2k(d+k)}{n+d}
\end{aligned}$$

# Sums of squares and Reznick's Positivstellensatz

# Hilbert's 17th problem

- $\mathbb{R}[x] \ni P(x) \geq 0 \iff P = Q_1(x)^2 + Q_2(x)^2$, for $Q_{1,2} \in \mathbb{R}[x]$
- $\mathrm{Pos}(d, n) := \{P \in \mathbb{R}[x_1, \ldots, x_d]$ hom. of deg. $2n$, $P(x) \geq 0, \forall x\}$
- $\mathrm{SOS}(d, n) := \{\sum_i Q_i^2$ with $Q_i \in \mathbb{R}[x_1, \ldots, x_d]$ hom. of deg. $n\}$
- Hilbert 1888:

$$\mathrm{SOS}(d, n) \subseteq \mathrm{Pos}(d, n), \text{ eq. iff } (d, n) \in \{(d, 1), (2, n), (3, 2)\}$$

- The Motzkin polynomial

$$M(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2$$

  is positive but not SOS
- Membership in SOS can be decided with a SDP: $P \in \mathrm{SOS}(d, n)$ iff $\exists A \geq 0$ such that $P = \langle v_{d,n} | A | v_{d,n} \rangle$, where $v_{d,n}$ is the vector containing all the hom. monomials in $d$ variables of degree $n$

# Reznick's Positivstellensatz

- Hilbert 1900, Artin 1927:

$$P \geq 0 \iff P = \sum_i \frac{Q_i^2}{R_i^2}$$

  In particular, if $P \geq 0$, there exists $R$ such that $R^2 P$ is SOS
- Polya 1928: $P$ even, $P \geq 0 \implies \exists r$ such that $(\sum_i x_i^2)^r P$ has non-negative coefficients (and thus is SOS)

**Theorem. [Reznick 1995]**

Let $P \in \mathrm{Pos}(d, k)$ such that $m(P) := \min_{\|x\|=1} P(x) > 0$. Then, for all

$$n \geq \frac{dk(2k-1)}{2\ln 2} \frac{M(P)}{m(P)} - \frac{d}{2}$$

we have

$$\|x\|^{2(n-k)} P(x) = \sum_{j=1}^{r} t_j \langle x, a_j \rangle^{2n},$$

where $t_j > 0$ and $a_j \in \mathbb{R}^d$

# A complex version of Reznick's PSS

- In the complex case, we are interested in bi-homogeneous polynomials of degree $n$ in $d$ complex variables: $P(z_1, \ldots, z_d)$ is hom. in the variables $z_i$ and also in $\bar{z}_i$.
- Bi-hom. polynomials are in one-to-one correspondence with operators on $\vee^n \mathbb{C}^d$:

$$P(z_1, \ldots, z_d) = \langle z^{\otimes n} | W | z^{\otimes n} \rangle$$

- Self-adjoint $W$ are associated to real, bi-hom. polynomials
- Non-negative polynomials $P$ are associated to block-positive matrices $W$:

$$\langle z^{\otimes n} | W | z^{\otimes n} \rangle \geq 0, \qquad \forall z \in \mathbb{C}^d$$

- $W$ PSD $\implies$ $P$ SOS: if $W = \sum_j t_j |a_j\rangle\langle a_j|$, then

$$P(z) = \sum_j t_j |\langle z^{\otimes n}, a_j \rangle|^2$$

- $\|z\|^{2n} = \langle z^{\otimes n} | P_{sym}^{(d,n)} | z^{\otimes n} \rangle$

# A complex version of Reznick's PSS

**Theorem.**

Consider $W = W^* \in \mathcal{B}(\vee^k \mathbb{C}^d \otimes \mathbb{C}^D)$ with $m(W) > 0$ and $k \geq 1$. Then, for any

$$n \geq \frac{dk(2k-1)}{\ln\left(1 + \frac{m(W)}{M(W)}\right)} - k \tag{1}$$

with $n \geq k$, we have

$$\|x\|^{2(n-k)} p_W(x,y) = \int p_{\tilde{W}}(\varphi, y) |\langle \varphi, x \rangle|^{2n} \mathrm{d}\varphi$$

with $p_{\tilde{W}}(\varphi, y) \geq 0$ for all $\varphi \in \mathbb{C}^d$ and $y \in \mathbb{C}^D$, where $p_{\tilde{W}}(\varphi, y)$ is a bihermitian form of degree $k$ in $\varphi$ and $\overline{\varphi}$ and degree 1 in $y$ and $\overline{y}$, explicitly computable in terms of $W$, and $\mathrm{d}\varphi$ is any $(n+k)$ spherical design. In the case $k = 1$, the bound (1) can be improved

$$n \geq d\frac{M(W)}{m(W)} - 1.$$

Similar result obtained by [To and Yeung] with worse bounds and in a less general setting, by "complexifying" Reznick's proof

# The proof: inverting the Chiribella formula

# Proof strategy

- The equality

$$\|x\|^{2(n-k)} p_W(x,y) = \int p_{\tilde{W}}(\varphi, y) |\langle \varphi, x \rangle|^{2n} \mathrm{d}\varphi$$

reads, in terms of linear maps over symmetric spaces

$$\mathrm{Clone}_{k\to n} \otimes \mathrm{id}_D = \left[\mathrm{MP}_{k\to n} \circ \tilde{\Psi}\right] \otimes \mathrm{id}_D$$

- The fact that the polynomial $p_{\tilde{W}}$ is non-negative reads

$$\tilde{W} := \tilde{\Psi}(W) \text{ is block-positive } \iff \langle z^{\otimes n} | \tilde{W} | z^{\otimes n} \rangle \geq 0$$

- Re-write the Chiribella identity as

$$\begin{aligned}
\mathrm{MP}_{n\to k} &= \sum_{s=0}^{k} c(n,k,s)\, \mathrm{Clone}_{s\to k} \circ \mathrm{Tr}_{n\to s} \\
&= \sum_{s=0}^{k} c(n,k,s)\, \mathrm{Clone}_{s\to k} \circ \mathrm{Tr}_{k\to s} \circ \mathrm{Tr}_{n\to k} \\
&= \Phi^{(n)}_{k\to k} \circ \mathrm{Tr}_{n\to k}
\end{aligned}$$

# Proof strategy

- $\mathrm{MP}_{n\to k} = \Phi^{(n)}_{k\to k} \circ \mathrm{Tr}_{n\to k}$

**Key fact.**

The linear map $\Phi^{(n)}_{k\to k} : \vee^k \mathbb{C}^d \to \vee^k \mathbb{C}^d$ is invertible, with inverse

$$\Psi^{(n)}_{k\to k} := \sum_{s=0}^{k} q(n,k,s)\, \mathrm{Clone}_{s\to k} \circ \mathrm{Tr}_{k\to s}$$

with

$$q(n,k,s) := (-1)^{s+k} \frac{\binom{n+s}{s}\binom{k}{s}}{\binom{n}{k}} \frac{d[k]}{d[s]}$$

- Hence, up to some constants, $\mathrm{Clone}_{k\to n} = \mathrm{MP}_{k\to n} \circ \Psi^{(n)}_{k\to k}$
- Final step: use hypotheses on $n, k, m(W), M(W)$ to ensure $\Psi^{(n)}_{k\to k}(W)$ is block-positive

# Proof strategy

Note: $p_{\mathrm{Tr}^*_{k\to n}(W)}(x) = \|x\|^{2(n-k)} p_W(x)$

**Lemma.**

For any $W \in \mathcal{B}(\vee^k \mathbb{C}^d)$, we have

$$p_{\mathrm{Tr}_{k\to k-s}(W)} = ((k)_s)^{-2} \Delta_{\mathbb{C}}^s p_W,$$

where $\Delta_{\mathbb{C}}$ is the Laplacian

$$\Delta_{\mathbb{C}} = \sum_{i=1}^{d} \frac{\partial^2}{\partial \bar{z}_i \partial z_i}$$

**Lemma.**

For any $W = W^* \in \mathcal{B}(\vee^k \mathbb{C}^d)$ we have

$$\forall \|z\| = 1, \qquad \left| (\Delta_{\mathbb{C}}^s p_W)(z) \right| \leq 4^{-s} (2d)^s (2k)_{2s} M(W)$$

# Proof strategy

- Assume, wlog, $D = 1$, i.e. there is no $y$

$$\begin{aligned} p_{\tilde{W}}(\varphi) &= \sum_{s=0}^{k} q(n,k,s) \langle \varphi^{\otimes k} | \, \mathrm{Clone}_{s\to k} \circ \mathrm{Tr}_{k\to s}(W) | \varphi^{\otimes k} \rangle \\ &= \sum_{s=0}^{k} q(n,k,s) \|\varphi\|^{2(k-s)} \langle \varphi^{\otimes s} | \, \mathrm{Tr}_{k\to s}(W) | \varphi^{\otimes s} \rangle \\ &= \sum_{s=0}^{k} q(n,k,s) \|\varphi\|^{2(k-s)} p_{\mathrm{Tr}_{k\to s}(W)}(\varphi) \\ &= \sum_{s=0}^{k} \hat{q}(n,k,s) \|\varphi\|^{2(k-s)} (\Delta_{\mathbb{C}}^{k-s} p_W)(\varphi) \end{aligned}$$

- Use the complex version of the Bernstein inequality

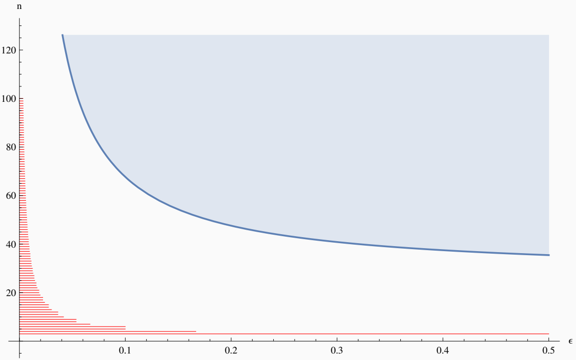
$$p_{\tilde{W}}(\varphi) \geq \left[ m(W) \tilde{q}(n,k,k) - M(W) \sum_{s=0}^{k-1} |\tilde{q}(n,k,s)| \right]$$

# How good are the bounds?

- Consider the modified Motzkin polynomial

$$p_\varepsilon(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2 + \varepsilon(x^2 + y^2 + z^2)$$

- We have $m(p_\varepsilon) = \varepsilon$; $M(p_\varepsilon) = \varepsilon + 4/27$
- Let $p_{n,\varepsilon}(x, y, z) := (x^2 + y^2 + z^2)^{n-3} p_\varepsilon(x, y, z)$. If a PSS decomposition holds, then the $[2p, 2q, 2r]$ coefficient of $p_{n,\varepsilon}$ must be positive $\rightsquigarrow$ lower bound on optimal $n$



# Thank you!

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