Random quantum states and channels

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Outline

Quantum states. Entanglement

Mixed quantum states

Random quantum states

Block-modified random states

Quantum states. Entanglement

Quantum states - the big picture

- One system

States	Deterministic	Random mixture
Classical	$x \in \{1, 2, \ldots, d\}$	$p \in \mathbb{R}^d,\ p_i \geq 0,\ \sum_i p_i = 1$
Quantum	$\psi \in \mathbb{C}^d,\ \|\psi\| = 1$	$\rho \in \mathcal{M}_d(\mathbb{C}),\ \rho \geq 0,\ \operatorname{Tr} \rho = 1$

- Two (or more) classical systems: cartesian product of individual systems
- Two (or more) quantum systems: tensor product of individual systems (at the level of Hilbert spaces or at the level of matrices)

$$\Downarrow$$

$$\boxed{\textit{entanglement}}$$

Axioms of Quantum Mechanics with pure states

- To every quantum mechanical system, we associate a Hilbert space $\mathcal{H} \cong \mathbb{C}^d$. The state of a system is described by a unit vector $|\psi\rangle \in \mathcal{H}$.

Example

The qubit - a two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$. States in superposition are allowed: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $\{|0\rangle, |1\rangle\}$ is an orthonormal basis of $\mathbb{C}^2$; we have $|\alpha|^2 + |\beta|^2 = 1$.

- States evolve according to unitary transformations $U \in \mathcal{U}(d)$: $|\psi\rangle \mapsto U|\psi\rangle$. Physically, $U = \exp(-itH)$ for an Hamiltonian H.
- Observable quantities correspond to Hermitian operators $A \in \mathcal{B}(\mathcal{H})$. Let $A = \sum_i \lambda_i P_i$ be the spectral decomposition of A. Born's rule asserts that, when measuring a quantum system in state $|\psi\rangle$,

$$\mathbb{P}[\text{ we observe } \lambda_i\,] = \langle\psi|P_i|\psi\rangle$$

and that, conditionally on observing λ_i, the system's state collapses to

$$|\psi'\rangle = \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}}.$$

Composite systems. Entanglement

For a system composed of two parts A (Alice, ) and B (Bob, ), with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the total Hilbert space is the tensor product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

A general two-qubit state $|\psi\rangle_{AB} \in \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$ is given by

$$|\psi\rangle_{AB} = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle,$$

where $|ij\rangle = |i\rangle \otimes |j\rangle$, and α_{ij} are complex amplitudes.

Definition

A pure state $|\psi\rangle_{AB}$ is called separable if $|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$.
Non-separable states are called entangled.

Entangled states are a key resource in quantum information, needed to obtain the computational speedups or to guarantee security of cryptographic protocols.

Separable states: $|\psi\rangle_{AB} = |00\rangle$ or $|\varphi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$
Entangled state: the Bell state $|\Omega\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Pure state entanglement is generic

Bipartite states can be seen as (rectangular matrices), via the isomorphism $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \cong \mathcal{M}_{d_A \times d_B}(\mathbb{C})$.

Proposition — Schmidt decomposition

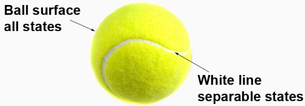
Given any quantum state $|\psi\rangle_{AB}$ there exist orthonormal families $\{|e_i\rangle\}_{i=1}^r \subseteq \mathbb{C}^{d_A}$, $\{|f_i\rangle\}_{i=1}^r \subseteq \mathbb{C}^{d_B}$ and a probability vector p such that

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{p_i}|e_i\rangle \otimes |f_i\rangle.$$

A state is separable iff $p = (1, 0, \dots, 0)$ iff the corresp. matrix is rank one. The Shannon entropy of p is called the entanglement entropy of $|\psi\rangle$

$$E(|\psi\rangle) = H(p) = -\sum p_i \log p_i.$$

All bi-partite quantum pure states have dimension $d_A d_B - 1$, whereas product states have dimension $d_A + d_B - 2$, which is strictly smaller $\implies$ a generic pure state is entangled!



Quantum entanglement for pure states

Separable pure states = rank 1 tensors

Entangled pure states = rank ≥ 2 tensors

Mixed quantum states

Mixed states and entanglement

Mixed quantum systems with d degrees of freedom are described by density matrices or mixed states

$$\rho \in \mathcal{M}^{1,+}(\mathbb{C}^d); \qquad \mathrm{Tr}\rho = 1 \text{ and } \rho \geq 0.$$

Pure states are the particular case of rank one projectors, and correspond to unit vectors $\psi \in \mathbb{C}^d$

$$|\psi\rangle\langle\psi| \in \mathcal{M}^{1,+}(\mathbb{C}^d).$$

They are the extreme points of the convex body $\mathcal{M}^{1,+}(\mathbb{C}^d)$.

Two quantum systems: $\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$.

A mixed state ρ_{AB} is called separable if it can be written as a convex combination of product states

$$\rho_{AB} \in \mathcal{SEP} \iff \rho_{AB} = \sum_i t_i \sigma_i^{(A)} \otimes \sigma_i^{(B)},$$

with $t_i \geq 0$, $\sum_i t_i = 1$, $\sigma_i^{(A,B)} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_{A,B}})$. Non-separable states are called entangled.

Mixed state entanglement is hard, but...

Deciding if a given ρ_{AB} is separable is NP-hard. Detecting entanglement for general states is a difficult, central problem in QIT.

A map $f : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^{d'})$ is called

- positive if $A \geq 0 \implies f(A) \geq 0$;
- completely positive if $\mathrm{id}_k \otimes f$ is positive for all $k \geq 1$.

If $f : \mathcal{M}(\mathbb{C}^{d_B}) \to \mathcal{M}(\mathbb{C}^{d_B})$ is CP, then for every state ρ_{AB} one has $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geq 0$.

If $f : \mathcal{M}(\mathbb{C}^{d_B}) \to \mathcal{M}(\mathbb{C}^{d_B})$ is only positive, then for every separable state ρ_{AB}, one has $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geq 0$.

Entanglement detection via positive, but not CP maps

- Positive, but not CP maps f yield entanglement criteria: given ρ_{AB}, if $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \not\geq 0$, then ρ_{AB} is entangled.
- The following converse holds: if, for all positive, but not CP maps f, $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geq 0$, then ρ_{AB} is separable.

- The transposition map $\Theta(X) = X^\top$ is positive, but not CP. Put

$$\mathcal{PPT} := \{\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \,|\, [\mathrm{id}_{d_A} \otimes \Theta_{d_B}](\rho_{AB}) \geq 0\}.$$

- The reduction map $R(X) = \mathrm{Tr}(X) \cdot I - X$ is positive, but not CP.

$$\mathcal{RED} := \{\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \,|\, [\mathrm{id}_{d_A} \otimes R_{d_B}](\rho_{AB}) \geq 0\}.$$

- Both criteria above detect pure entanglement: for $f = \Theta, R$,

$$[\mathrm{id}_{d_A} \otimes f](|\psi\rangle_{AB}\langle\psi|) \geq 0 \iff |\psi\rangle_{AB} \text{ is separable.}$$

The PPT criterion at work

- Recall the Bell state $\rho_{12} = |\psi\rangle\langle\psi|$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B).$$

- Written as a matrix in $\mathcal{M}^{1,+}_{2\cdot 2}(\mathbb{C})$

$$\rho_{AB} = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) = \frac{1}{2}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

- Partial transposition: transpose each block B_{ij}:

$$[\mathrm{id}_2 \otimes \Theta](\rho_{AB}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- This matrix is no longer positive $\implies$ the state is entangled.

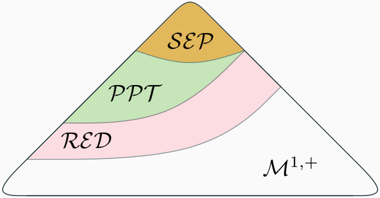
The problem we consider

$\mathcal{M}^{1,+} = \{\rho : \mathrm{Tr}\rho = 1 \text{ and } \rho \geq 0\}$

$\mathcal{SEP} = \left\{ \sum_i t_i \rho_i^{(A)} \otimes \rho_i^{(B)} \right\}$

$\mathcal{PPT} = \{\rho_{AB} : [\mathrm{id}_{d_A} \otimes \Theta_{d_B}](\rho_{AB}) \geq 0\}$

$\mathcal{RED} = \{\rho_{AB} : [\mathrm{id}_{d_A} \otimes R_{d_B}](\rho_{AB}) \geq 0\}$



Problem

Compare the convex sets

$$\mathcal{SEP} \subseteq \mathcal{PPT} \subseteq \mathcal{RED} \subseteq \mathcal{M}^{1,+}(\mathbb{C}^{d_A d_B}).$$

- For $(d_A, d_B) \in \{(2,2), (2,3), (3,2)\}$ we have $\mathcal{SEP} = \mathcal{PPT}$. In other dimensions, the inclusion $\mathcal{SEP} \subset \mathcal{PPT}$ is strict.
- For $d_B = 2$ we have $\mathcal{PPT} = \mathcal{RED}$. In other dimensions, the inclusion $\mathcal{PPT} \subset \mathcal{RED}$ is strict.

Random quantum states

Probability measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- We want to measure volumes of subsets of $\mathcal{M}_d^{1,+}(\mathbb{C})$, with $d = d_A d_B$.
- A natural choice is to use the Lebesgue measure (see $\mathcal{M}_d^{1,+}(\mathbb{C})$ as a compact subset of $\mathcal{M}_d^{sa}(\mathbb{C})$). The set of separable states $\mathcal{SEP}$ has positive volume, since $\mathcal{SEP}$ contains an open ball around I/d.
- Another choice - open quantum systems point of view: assume your system Hilbert space $\mathbb{C}^d = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ is coupled to an environment $\mathbb{C}^{d_C}$.
- On the tri-partite system $\mathcal{H}_{ABC} = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_C}$, consider a random pure state $|\psi\rangle_{ABC}$, i.e. a uniform random point on the unit sphere of the total Hilbert space $\mathcal{H}_{ABC}$.
- Trace out the environment $\mathbb{C}^{d_C}$ to get a *random density matrix*

$$\rho_{AB} = \mathrm{Tr}_C |\psi\rangle_{ABC}\langle\psi|.$$

- These probability measures have been introduced by Życzkowski and Sommers and they are called the induced measures of parameters $d = d_A d_B$ and $s = d_C$; we denote them by $\mu_{d,s}$.
- Remarkably, the Lebesgue measure is obtained for $s = d$.

Probability measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- Here's an equivalent way of defining the measures $\mu_{d,s}$, in the spirit of Random Matrix Theory.
- Let $X \in \mathcal{M}_{d \times s}(\mathbb{C})$ be a $d \times s$ matrix with i.i.d. complex standard Gaussian entries (i.e. a Ginibre random matrix). Define

$$W_{d,s} = XX^* \text{ and } \mathcal{M}^{1,+}(\mathbb{C}^d) \ni \rho_{d,s} = \frac{XX^*}{\mathrm{Tr}(XX^*)} = \frac{W_{d,s}}{\mathrm{Tr}\, W_{d,s}}.$$

- The random matrix $W_{d,s}$ is called a Wishart matrix and the distribution of $\rho_{d,s}$ is precisely $\mu_{d,s}$.
- The measure $\mu_{d,s}$ is unitarily invariant: if $\rho \sim \mu_{d,s}$ and U is a fixed unitary matrix, then $U\rho U^* \sim \mu_{d,s}$.
- Density of $\mu_{d,s}$: $\mathrm{d}\mathbb{P}(\rho) = C_{d,s} \det(\rho)^{s-d} \mathbf{1}_{\rho \geq 0, \, \mathrm{Tr}\, \rho = 1} \, \mathrm{d}\rho$.
- Integrating out the eigenvectors, we obtain the eigenvalue density formula for random quantum states:

$$\mathrm{d}\mathbb{P}(\lambda_1, \ldots, \lambda_d) = C'_{d,s} \left[\prod_i \lambda_i^{s-d} \right] \left[\prod_{i<j} (\lambda_i - \lambda_j)^2 \right] \mathbf{1}_{\lambda_i \geq 0, \sum_i \lambda_i = 1} \, \mathrm{d}\lambda.$$

Eigenvalues for induced measures

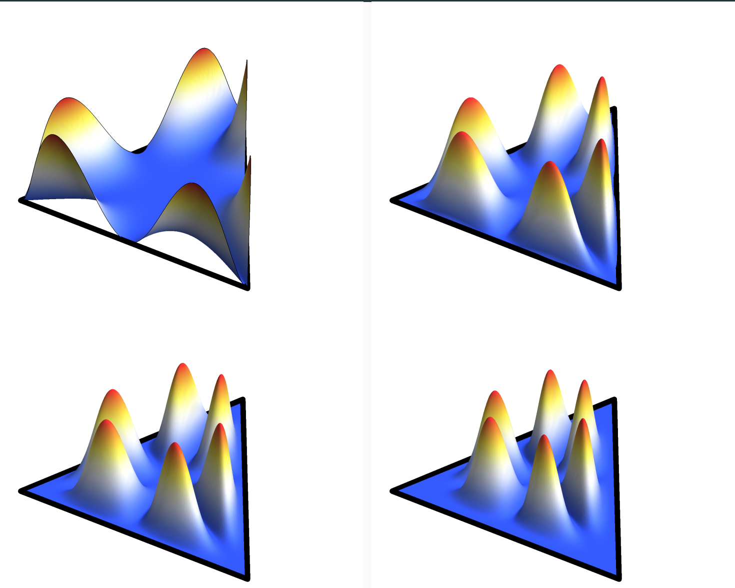


Figure 1: Induced measures for $d = 3$ and $s = 3, 5, 7, 10$.

Eigenvalues for induced measures

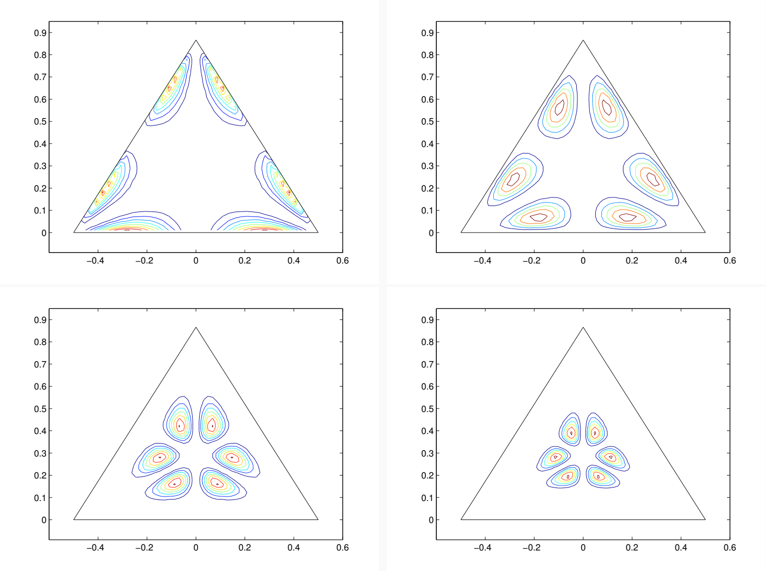


Figure 2: Induced measures for $d = 3$ and $s = 3, 5, 7, 10$.

Volume of convex sets under the induced measures

- Fix d, and let $C \subset \mathcal{M}^{1,+}(\mathbb{C}^d)$ a convex body, with $\mathrm{I}_d/d \in \mathrm{int}(C)$. Then

$$\lim_{s\to\infty} \mu_{d,s}(C) = 1.$$

In other words, the eigenvalues of a random density matrix $\rho_{AB} \sim \mu_{d,s}$ with d fixed and $s \to \infty$ converge to $1/d$.

Definition

A pair of functions $(s_0(d), s_1(d))$ are called a threshold for a family of convex sets $(K_d)_d$ if both conditions below hold

If $s(d) \lesssim s_0(d)$, then

$$\lim_{d\to\infty} \mu_{d,s(d)}(K_d) = 0;$$

If $s(d) \gtrsim s_1(d)$, then

$$\lim_{d\to\infty} \mu_{d,s(d)}(K_d) = 1.$$

Thresholds for entanglement criteria

- Below, the threshold functions $s_{0,1}(d)$ are of the form

$$s_0(d) = s_1(d) = cd; \qquad \text{we put } r := \min(d_A, d_B).$$

Crit. \ Reg.	$d_A = d_B \to \infty$	$d_B \to \infty$	$d_A \to \infty$
$\mathcal{SEP}$	$\infty \ (r \lesssim c \lesssim r \log^2 r)$	?	?
$\mathcal{PPT}$	4	$2 + 2\sqrt{1 - \frac{1}{r^2}}$	$2 + 2\sqrt{1 - \frac{1}{r^2}}$
$\mathcal{RED}$	0	0	$\frac{(1+\sqrt{r+1})^2}{r(r-1)}$

- The results in the table above can be interpreted in the following way: for a convex set K having a threshold c, a random density matrix $\rho_{AB} \sim \mu_{d,s}$ with large s, d will satisfy
  - If $s/d > c$, $\mathbb{P}[\rho_{AB} \in K] \approx 1$
  - If $s/d < c$, $\mathbb{P}[\rho_{AB} \in K] \approx 0$.

Proof elements

- The main task is to compute the probability that some random matrices are positive semidefinite or not.
- This is a very difficult computation to perform at fixed Hilbert space dimension; the asymptotic theory is much easier (one or both $d_{A,B} \to \infty$).
- To a selfadjoint matrix $X \in \mathcal{M}_d(\mathbb{C})$, with spectrum $x = (x_1, \ldots, x_d)$, associate its empirical spectral distribution

$$\mu_X = \frac{1}{d} \sum_{i=1}^{d} \delta_{x_i}.$$

- The probability measure μ_X contains all the information about the spectrum of X.
- A sequence of matrices X_d converges in moments towards a probability measure μ if, for all integer $p \geq 1$,

$$\lim_{d \to \infty} \frac{1}{d} \mathrm{Tr}(X_d^p) = \lim_{d \to \infty} \int x^p d\mu_{X_d}(x) = \int x^p d\mu(x).$$

Wishart matrices

Theorem (Marcenko-Pastur)

Let W be a complex Wishart matrix of parameters (d, cd). Then, almost surely with $d \to \infty$, the empirical spectral distribution of W/d converges in moments to a free Poisson distribution (a.k.a. Marčenko-Pastur distribution) π_c of parameter c.

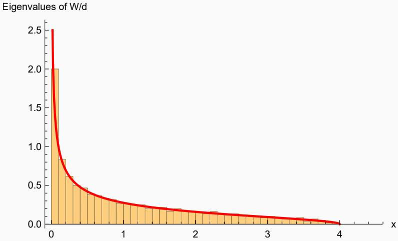


Figure 3: Eigenvalue distribution for Wishart matrices. In blue, the density of theoretical limiting distribution, π_c. In the two pictures, $d = 1000$, and $c = 1, 5$.

Partial transposition of a Wishart matrix

Theorem (Banica, N.)

Let W be a complex Wishart matrix of parameters (dn, cdn). Then, almost surely with $d \to \infty$, the empirical spectral distribution of $[\mathrm{id} \otimes \Theta](W_{AB}/d)$ converges in moments to a free difference of free Poisson distributions of respective parameters $cn(n \pm 1)/2$.

Corollary

The limiting measure above has positive support iff

$$c > c_{PPT} := 2 + 2\sqrt{1 - \frac{1}{n^2}}.$$

Partial transposition criterion - numerics

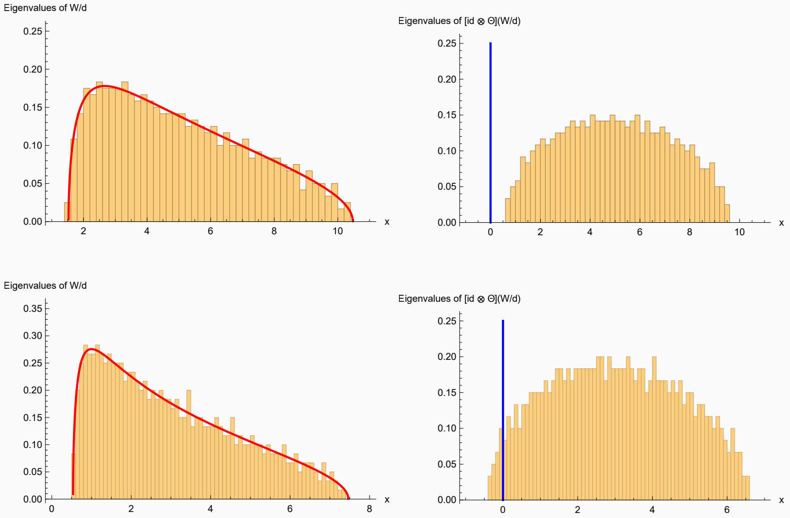


Figure 4: Wishart matrices before (left) and after (right) the application of the partial transposition. Here, $d = d_A = 200$, $n = d_B = 3$, and $c = 5$ (top), $c = 3$ (bottom). Note that $5 > c_{PPT} = 3.88562 > 3$.

Reduction of a Wishart matrix

Theorem (Jivulescu, Lupa, N.)

Let W be a complex Wishart matrix of parameters (dn, cdn). Then, almost surely with $d \to \infty$, the empirical spectral distribution of $[\mathrm{id} \otimes R](W_{AB}/d)$ converges in moments to a compound free Poisson distribution $\pi_{\nu_{n,c}}$ of parameter $\nu_{n,c} = c\delta_{1-n} + c(n^2-1)\delta_1$.

Corollary

The limiting measure above has positive support iff

$$c > c_{RED} := \frac{(1+\sqrt{n+1})^2}{n(n-1)}.$$

Remark

We have, for $n = 2$, $c_{PPT} = c_{RED} = 2 + \sqrt{3}$: the two criteria are know to be equivalent for qubit-qudit systems. For $n \geq 3$, we have $c_{PPT} > c_{RED}$: the reduction criterion is, in general, weaker than the PPT criterion.

Reduction criterion - numerics

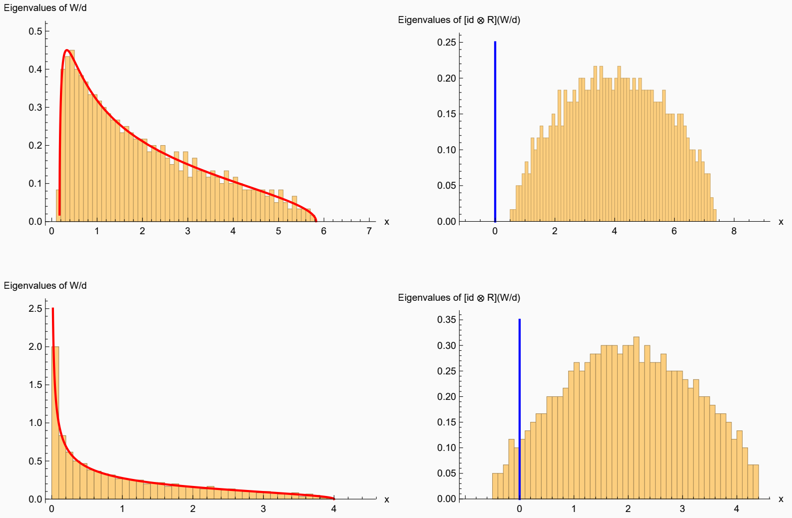


Figure 5: Wishart matrices before (left) and after (right) the application of the partial reduction map. Here, $d = d_A = 200$, $n = d_B = 3$, and $c = 2$ (top), $c = 1$ (bottom). Note that $2 > c_{RED} = 1.5 > 1$.

The free additive convolution of probability measures

- Given two self-adjoint matrices X, Y with spectra x, y, what is the spectrum of $X + Y$?
- In general, a very difficult problem, the answer depends on the relative position of the eigenspaces of X and Y (Horn problem).
- When the size of X, Y is large, and the eigenvectors are in general position, free probability theory gives the answer.
- Free additive convolution of two compactly supported probability distributions μ, ν: sample $x, y \in \mathbb{R}^d$ from μ, ν and consider

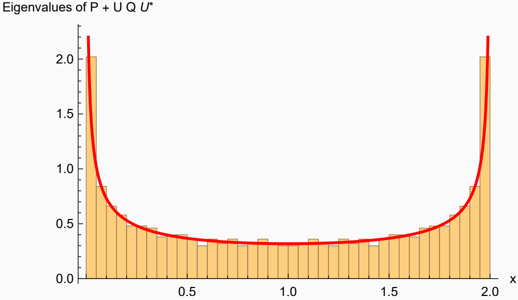
$$Z := \mathrm{diag}(x) + U\mathrm{diag}(y)U^*,$$

  where U is a $d \times d$ Haar unitary random matrix. Then, as $d \to \infty$, the empirical eigenvalue distribution of Z converges to a probability measure denoted by $\mu \boxplus \nu$.
- The operation $\boxplus$ is called free additive convolution, and it can be computed via the $\mathcal{R}$-transform (a kind of Fourier transform in the free world)

Free additive convolution - an example

- We have

$$\left[\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right] \boxplus \left[\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right] = \frac{1}{\pi\sqrt{x(2-x)}}\mathbf{1}_{(0,2)}(x)\,\mathrm{d}x.$$



- Compare to the classical situation, where $*$ denotes the (additive) classical convolution

$$\left[\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right] * \left[\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right] = \frac{1}{4}\delta_0 + \frac{1}{2}\delta_1 + \frac{1}{4}\delta_2.$$

The free Poisson distribution

- The limiting distribution of Wishart matrices (and of random density matrices from $\mu_{d,cd}$) is the free Poisson distribution

$$\pi_c := \max(1-c,0)\delta_0 + \frac{\sqrt{4c-(x-1-c)^2}}{2\pi x}\mathbf{1}_{[(1-\sqrt{c})^2,(1+\sqrt{c})^2]}(x)\,\mathrm{d}x.$$

- One can show a free Poisson Central Limit Theorem:

$$\lim_{n\to\infty}\left[\left(1-\frac{c}{n}\right)\delta_0+\frac{c}{n}\delta_1\right]^{\boxplus n}=\pi_c.$$

- The limit measure for $[\mathrm{id}\otimes\Theta](W_{AB}/d)$ is

$$\pi_c^{PPT} := \pi_{cn(n+1)/2}\boxplus D_{-1}\pi_{cn(n-1)/2}.$$

- The free compound Poisson measure of parameter ν is defined via a generalized free Poisson central limit theorem

$$\lim_{n\to\infty}\left[\left(1-\frac{\nu(\mathbb{R})}{n}\right)\delta_0+\frac{1}{n}\nu\right]^{\boxplus n}=:\pi_\nu.$$

- The limit measure for $[\mathrm{id}\otimes R](W_{AB}/d)$ is

$$\pi_c^{RED} := \pi_{c\delta_{1-n}+c(n^2-1)\delta_1}.$$

Block-modified random states

Recap: how powerful are the entanglement criteria?

- Let $f : \mathbb{M}_m \to \mathbb{M}_n$ be a given positive linear map (usually, f not CP).
- If $[f \otimes \mathrm{id}](\rho) \not\geq 0$, then $\rho \in \mathbb{M}_m \otimes \mathbb{M}_d$ is entangled.
- If $[f \otimes \mathrm{id}](\rho) \geq 0$, then ... we do not know.
- Define

$$\mathcal{K}_f := \{\rho : [f \otimes \mathrm{id}](\rho) \geq 0\} \supseteq \mathcal{SEP}.$$

- We would like to compare (e.g. using the volume) the sets $\mathcal{K}_f$ and $\mathcal{SEP}$.
- Several probability measures on the set $\mathbb{M}_{md}^{1,+}$: for any parameter $s \geq md$, let W be a Wishart matrix of parameters (md, s): $W = XX^*$, with $X \in \mathbb{M}_{md \times s}$ a Ginibre random matrix (the entries of X are i.i.d. complex Gaussian random variables).
- Let $\mathbb{P}_s$ be the probability measure obtained by pushing forward the Wishart measure by the map $W \mapsto W/\mathrm{Tr}(W)$.
- To compute $\mathbb{P}_s(\mathcal{K}_f)$, one needs to decide whether the spectrum of the random matrix $[f \otimes \mathrm{id}](W)$ is positive (here, d is large, m, n are fixed) $\rightsquigarrow$ block modified matrices.

Block-modified random matrices - previous results

Many cases studied independently, using the method of moments for Wishart matrices; no unified approach, each case requires a separate analysis:

- [Aubrun '12]: the asymptotic spectrum of $W^\Gamma := [\mathrm{id} \otimes \mathrm{t}](W)$ is a shifted semicircular, for $W \in \mathbb{M}_d \otimes \mathbb{M}_d$, $d \to \infty$
- [Banica, N. '13]: the asymptotic spectrum of $W^\Gamma := [\mathrm{id} \otimes \mathrm{t}](W)$ is a free difference of free Poisson distributions, for $W \in \mathbb{M}_m \otimes \mathbb{M}_d$, $d \to \infty$, m fixed
- [Banica, N. '15]: the asymptotic spectrum of $W^f := [\mathrm{id} \otimes \mathrm{f}](W)$ is the free multiplicative convolution between a free compound Poisson distribution and the distribution of $f(I)$; the result requires f to come from a "wire diagram"
- [Jivulescu, Lupa, N. '14,'15]: the asymptotic spectrum of $W^{red} := W - [\mathrm{Tr} \otimes \mathrm{id}](W) \otimes I$ is a compound free Poisson distribution, for $W \in \mathbb{M}_m \otimes \mathbb{M}_d$, $d \to \infty$, m fixed (here, $f(X) = X - \mathrm{Tr}(X) \cdot I$)
- etc...

The problem

- Consider a sequence of unitarily invariant random matrices $X_d \in \mathbb{M}_n \otimes \mathbb{M}_d$:

$$\forall U \in \mathcal{U}_{nd}, \quad \mathrm{law}(X_d) = \mathrm{law}(U X_d U^*).$$

- Fix n and assume that, as $d \to \infty$, the matrices X_d have have limiting spectral distribution μ:

$$\lim_{d \to \infty} \frac{1}{nd} \sum_{i=1}^{nd} \delta_{\lambda_i(X_d)} = \mu.$$

- Define the modified version of X_d:

$$X_d^f = [f \otimes \mathrm{id}_d](X_d).$$

- Our goal: compute μ^f, the limiting spectral distribution of X_d^f, as a function of
  1. The initial distribution μ
  2. The function f.

- Results: achieved this for all μ and a fairly large class of f.
- Tools: operator-valued free probability theory.

Taking the limit

- We can write

$$X_d^f = [f \otimes \mathrm{id}](X_d) = \sum_{i,j,k,l=1}^{n} c_{ijkl}(E_{ij} \otimes I_d) X_d (E_{kl} \otimes I_d) \in \mathbb{M}_n \otimes \mathbb{M}_d,$$

  for some coefficients $c_{ijkl} \in \mathbb{C}$, which are actually the entries of the Choi matrix of f (see tomorrow's talk).

- At the limit:

$$x^f = \sum_{i,j,k,l=1}^{n} c_{ijkl} e_{i,j} x e_{k,l},$$

  for some random variable x having the same distribution as the limit of X_d and some (abstract) matrix units e_{ij}.

$\rightsquigarrow$ In the rectangular case $m \neq n$, one needs to use the techniques of Benaych-Georges; we will have freeness with amalgamation on $\langle p_m, p_m \rangle$.

The limiting distributions of block-modified matrices

Theorem

For "well-behaved" functions f, then the distribution of x^f has the following R-transform:

$$R_{x^f}(z) = \sum_{i=1}^{s} d_i \rho_i R_x \left[\frac{\rho_i}{n} z \right],$$

where ρ_i are the distinct eigenvalues of C and nd_i are ranks of the corresponding eigenprojectors. In other words, if μ, resp. μ^f, are the respective distributions of x and x^f, then

$$\mu^f = \boxplus_{i=1}^{s} (D_{\rho_i/n}\mu)^{\boxplus nd_i}.$$

The transposition, $f(X) = X^\top$:

$$\mu^T = \left(D_{1/n}\mu^{\boxplus n(n+1)/2} \right) \boxplus \left(D_{-1/n}\mu^{\boxplus n(n-1)/2} \right).$$

Range of applications

The following functions are well behaved

1. Unitary conjugations $f(X) = UXU^*$
2. The trace and its dual $f(X) = \mathrm{Tr}(X)$, $f(x) = xI_n$
3. The transposition $f(X) = X^\top$
4. The reduction map $f(X) = I_n \cdot \mathrm{Tr}(X) - X$
5. Linear combinations of the above $f(X) = \alpha X + \beta \mathrm{Tr}(X) I_n + \gamma X^\top$
6. Mixtures of orthogonal automorphisms

$$f(X) = \sum_{i=1}^{n^2} \alpha_i U_i X U_i^*,$$

   for orthogonal unitary operators U_i

$$\mathrm{Tr}(U_i U_j^*) = n\delta_{ij}.$$

7. The Choi map

$$f([x_{ij}]) = \begin{bmatrix} ax_{11} + bx_{22} + cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11} + ax_{22} + bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11} + cx_{22} + ax_{33} \end{bmatrix}.$$

Support of the resulting measures

- Recall that we are interested ultimately in the positivity of the support of the resulting operators x^f
- It is in general hard to obtain analytical expressions for the support of x^f: one has to solve polynomials equations of large degree.
- Example: $\pi_c^{\mathrm{t}_\mathrm{n}}$ has positive support iff $c > 2 + 2\sqrt{1 - \frac{1}{n^2}}$

Lemma (Collins, Fukuda, Zhong '15)

Let μ be a probability measure having mean m and variance σ^2, whose support is contained in $[A, B]$. Then, for any $T \geq 1$ such that $\mu^{\boxplus T}$ has no atoms, we have

$$\mathrm{supp}(\mu^{\boxplus T}) \subseteq [A + m(T-1) - 2\sigma\sqrt{T-1}, B + m(T-1) + 2\sigma\sqrt{T-1}].$$

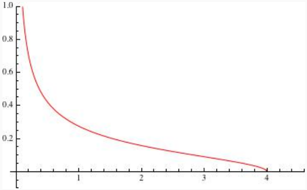
Proposition (I.N. '18)

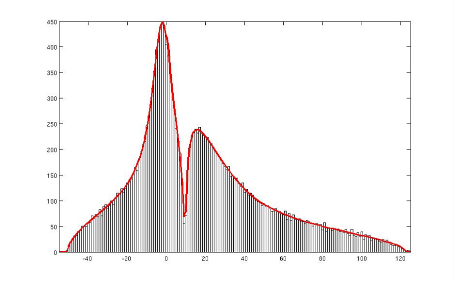
Let μ be a non-atomic probability measure having mean m and variance σ^2, whose support is contained in the compact interval $[A, B]$. Then, provided that $n(m - 2\sigma) > B - A + 2\sigma$, we have $\mathrm{supp}(\mu^\Gamma) \subset (0, \infty)$.

Marchenko-Pastur distribution

$$d\mu(x) = \frac{\sqrt{x(4-x)}}{2\pi x}\mathbf{1}_{(0,4]}(x)\,dx$$

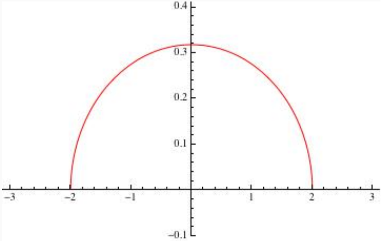
$$f\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 11a_{11} + 15a_{22} - 25a_{12} - 25a_{21} & 36a_{21} \\ 36a_{12} & 11a_{11} - 4a_{22} \end{bmatrix}$$

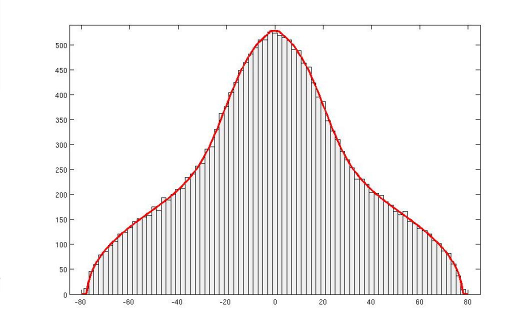




Wigner semicircle distribution

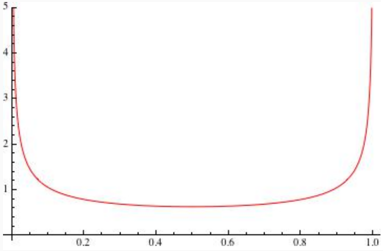
$$d\mu(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbf{1}_{[-2,2]}(x)\,dx.$$

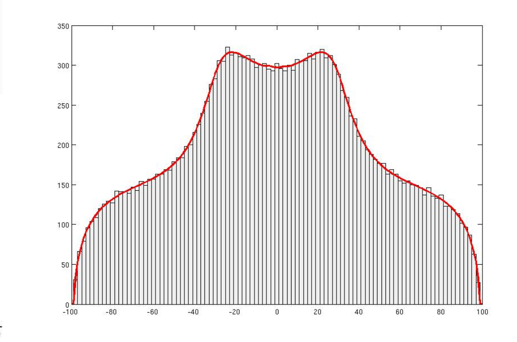




Arcsine distribution

$$d\mu(x) = \frac{1}{\pi\sqrt{x(1-x)}}\mathbf{1}_{(0,1)}(x)\,dx.$$





Merci!