Random quantum states and channels

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Summary from yesterday

- Pure quantum states of one particle: unit norm vectors inside a Hilbert space
- More particles $\rightsquigarrow$ take the tensor product of the Hilbert spaces
- Separable states = rank-1 tensors; entangled states = rank ≥ 2 tensors
- Random pure states: uniform point on the unit sphere
- Mixed quantum states (or density matrices): positive semidefinite matrices of unit trace $\rho \geq 0$, $\operatorname{Tr} \rho = 1$
- Extreme points of the set of mixed states = P_x, with x pure
- Separable (i.e. non-entangled state)

$$\rho_{AB} = \sum_i t_i \sigma_i^{(A)} \otimes \sigma_i^{(B)}$$

- Random mixed states: normalized Wishart matrix
- Partial transposition $\rho_{AB}^\Gamma := [\mathrm{id} \otimes \Theta](\rho_{AB})$. If $\rho_{AB}^\Gamma \not\geq 0$, then ρ_{AB} is entangled.

Outline

Quantum channels

Random quantum channels

Entanglement of subspaces

Random subspaces

Quantum channels

Quantum channels

Channels	Deterministic	Random mixture
Classical	$f : \{1, \ldots, d\} \to \{1, \ldots, d\}$	Q Markov (stochastic)
Quantum	$U \in \mathcal{U}(d)$	Φ CPTP map

- Classical channels (acting on probability vectors):
  - Positivity: for all i, j, $Q_{ij} \geq 0$
  - Mass preservation: for all j, $\sum_i Q_{ij} = 1$.
- Quantum channels: CPTP maps $\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_{d'}(\mathbb{C})$
  - CP - complete positivity: $\Phi \otimes \mathrm{id}_r$ is a positive map, $\forall r \geq 1$
  - TP - trace preservation: $\mathrm{Tr} \circ \Phi = \mathrm{Tr}$.

Structure of quantum channels

Theorem [Stinespring-Kraus-Choi]

Let $\Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ be a linear map. TFAE:

1. The map Φ is completely positive and trace preserving (CPTP).
2. [Stinespring] There exist an integer n ($n = d^2$ suffices) and an isometry $W : \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^n$ such that
$$\Phi(X) = [\mathrm{id}_d \otimes \mathrm{Tr}_n](WXW^*).$$
3. [Kraus] There exist operators $A_1, \ldots, A_n \in \mathcal{M}_d(\mathbb{C})$ satisfying $\sum_i A_i^* A_i = I_d$ such that
$$\Phi(X) = \sum_{i=1}^{n} A_i X A_i^*.$$
4. [Choi] The Choi matrix C_Φ is positive semidefinite, where
$$C_\Phi := \sum_{i,j=1}^{d} E_{ij} \otimes \Phi(E_{ij}) \in \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$$
and $[\mathrm{id} \otimes \mathrm{Tr}](C_\Phi) = I_d$.

Examples and non-examples

- The identity channel $\mathrm{id} : \mathcal{M}_d \to \mathcal{M}_d$ has the (un-normalized) Bell state as its Choi matrix

$$C_{\mathrm{id}} = \sum_{i,j=1}^{d} |ii\rangle\langle jj| = \sum_{i,j=1}^{d} e_i \otimes e_i \cdot e_j^* \otimes e_j^*.$$

- The totally depolarizing channel (or the conditional expectation on scalars) $\Delta(X) = (\operatorname{Tr} X)I/d$ has Choi matrix I_{d^2}/d
- The totally dephasing channel (or the conditional expectation on diagonal matrices) D has Kraus decomposition

$$D(\rho) = \sum_{i=1}^{d} |i\rangle\langle i|\rho|i\rangle\langle i|.$$

- The transposition $\Theta(\rho) = \rho^\top$ is **not** a quantum channel, since it is not completely positive. Its Choi matrix is $C_\Theta = F$, where F is the flip operator $Fx \otimes y = y \otimes x$. F has eigenvalues $+1$ with multiplicity $d(d+1)/2$ and -1 with multiplicity $d(d-1)/2$.

Intermezzo: block modified random matrices

- Consider a sequence of unitarily invariant random matrices $X_d \in \mathbb{M}_n \otimes \mathbb{M}_d$. Fix n and assume that, as $d \to \infty$, the matrices X_d have have limiting spectral distribution μ: $\lim_{d\to\infty} \frac{1}{nd} \sum_{i=1}^{nd} \delta_{\lambda_i(X_d)} = \mu$.
- Define the modified version of X_d:

$$X_d^f = [f \otimes \mathrm{id}_d](X_d).$$

- Our goal: compute μ^f, the limiting spectral distribution of X_d^f, as a function of the initial distribution μ and the linear functional f.

Theorem (Arizmendi, Vargas, N. '16)

If the Choi matrix C_f satisfies the unitarity condition, then

$$\mu^f = \boxplus_{i=1}^{s} (D_{\rho_i/n}\mu)^{\boxplus nd_i},$$

where ρ_i are the distinct eigenvalues of C_f and nd_i are ranks of the corresponding eigenprojectors P_i.

Unitarity condition: $\forall i$, $[\mathrm{id} \otimes \mathrm{Tr}](P_i) = d_i I_n$.

Random quantum channels

Definition

There exist several natural candidates for probability distributions on the convex set of quantum channels:

- The Lebesgue measure
- Pick the isometry W in the Stinespring decomposition at random: W is a Haar-random isometry
- Pick the Kraus operators A_i at random: G_i are i.i.d. Ginibre matrices, define $A_i = G_i S^{-1/2}$, with $S = \sum_i G_i^* G_i$
- Pick the Choi matrix at random: $\tilde{C}$ is a Wishart matrix, define $C := [I \otimes T^{-1/2}]\tilde{C}[I \otimes T^{-1/2}]^*$, with $T = [\mathrm{Tr} \otimes \mathrm{id}]\tilde{C}$.

Theorem (Kukulski, N., Pawela, Puchala, Zyczkowski '19)

The above distributions are identical, when the respective parameters match.

Computationally, the random Kraus operators procedure is the cheapest; mathematically, the random isometry procedure is the more interesting and easier to deal with.

More on the distribution of random quantum channels

- For channels $\Phi : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$, if s is an integer parameter, then:
  - One has to take a Haar-random isometry $W; \mathbb{C}^{d_1} \to \mathbb{C}^{d_2} \otimes \mathbb{C}^s$
  - One has to take G_i i.i.d. Ginibre matrices of size $d_2 \times d_1$, for $i = 1, 2, \ldots, s$
  - One has to take the un-normalized Choi matrix $\tilde{C}$ a Wishart matrix of parameters $d_1 d_2, s$
- The density of the normalized Choi matrix reads

$$f(C) = \delta([\mathrm{id} \otimes \mathrm{Tr}](C) - I_{d_1}) \det C^{s - d_1 d_2} \mathrm{dLeb}.$$

- The Lebesgue measure is obtained for $s = d_1 d_2$.
- For any fixed pure state $P_x = xx^*$, the output matrix $\rho = \Phi(P_x)$ follows the induced distribution of parameters (d_2, s), i.e. has the distribution of a trace-normalized Wishart.
- However, different inputs yield correlated outputs!

Some notions of entropy

- Let $\Delta_k = \{\lambda \in \mathbb{R}^k \, : \, \lambda_i \geq 0, \, \sum_i \lambda_i = 1, \}$ be the probability simplex. We write $\Delta_k^{\downarrow}$ for the set of ordered probability vectors, $\lambda_1 \geq \cdots \geq \lambda_k$.
- The Shannon entropy of a probability vector $\lambda \in \Delta_k$

$$H(\lambda) = -\sum_{i=1}^{k} \lambda_i \log \lambda_i \in [0, \log k].$$

- The von Neumann entropy of $\rho \in \mathcal{M}_k^{1,+}$

$$H(\rho) = -\mathrm{Tr}(\rho \log \rho) = -\sum_{i=1}^{k} \lambda_i(\rho) \log \lambda_i(\rho).$$

- For $p \geq 0$, define the p-Rényi entropy

$$H_p(\rho) = \frac{\log \mathrm{Tr}(\rho^p)}{1-p} = \frac{\log \sum_i \lambda_i(\rho)^p}{1-p}; \qquad H(\cdot) = \lim_{p \to 1} H_p(\cdot).$$

- The entropy is additive: $H_p(\rho_1 \otimes \rho_2) = H_p(\rho_1) + H_p(\rho_2)$.

Additivity of the minimum output entropy

The minimum output entropy of a quantum channel Φ is

$$\boxed{H_p^{\min}(\Phi) := \min_{\rho \in \mathcal{M}_d^{1,+}} H_p(\Phi(\rho)).}$$

Conjecture (Amosov, Holevo and Werner '00)

The quantity $H_p^{\min}$ is additive: for any quantum channels Φ_1, Φ_2

$$H_p^{min}(\Phi_1 \otimes \Phi_2) = H_p^{min}(\Phi_1) + H_p^{min}(\Phi_2).$$

- Additivity of $H_{p=1}^{\min}$ implies a simple formula for the capacity of channels to transmit classical information; in particular, it implies the additivity of the classical capacity C.
- $C(\Phi) = \lim_{r\to\infty} \frac{\chi(\Phi^{\otimes r})}{r}$, with the Holevo quantity

$$\chi(\Phi) := \max_{p_i, \rho_i} H\Big(\sum_i p_i \Phi(\rho_i)\Big) - \sum_i p_i H(\Phi(\rho_i))$$

- Compare with the capacity of a classical channel Q:

$$C(Q) = \min_X I(X : Y) \qquad \text{where } Y = Q(X).$$

Additivity of the minimum output entropy

Conjecture (Amosov, Holevo and Werner '00)

The quantity $H_p^{\min}(\Phi) = \min_{\rho \in \mathcal{M}_d^{1,+}} H_p(\Phi(\rho))$ *is additive: for any quantum channels* Φ_1, Φ_2

$$H_p^{min}(\Phi_1 \otimes \Phi_2) = H_p^{min}(\Phi_1) + H_p^{min}(\Phi_2).$$

- Given Φ_1, Φ_2, the $\leq$ direction of the equality is trivial, take $\rho_{12} = \rho_1 \otimes \rho_2$.
- Additivity has been shown to hold for a large class of channels: unitary, unital qubit, depolarizing, dephasing, entanglement breaking, ...
- But... the Additivity Conjecture is false ! [Hayden, Winter '08 for $p > 1$, Hastings '09 for $p = 1$]
- Counterexamples: mostly random channels. Deterministic counterexamples: '02 Werner & Holevo ($p > 4.79$), '07 Cubitt et al ($p < 0.11$) and '09 Grudka et al ($p > 2$).

Stinespring dilation

Theorem (Stinespring dilation)

For any channel $: \mathcal{M}_d \to \mathcal{M}_k$ *there exists an isometry* $W : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n$ *such that*

$$\Phi(\rho) = [\mathrm{id}_k \otimes \mathrm{Tr}_n](W \rho W^*).$$

- By convexity properties, the minimum output entropy of Φ is attained on pure states i.e. rank one projectors $P_x = xx^* = |x\rangle\langle x|$.
- Since $\Phi(P_x) = [\mathrm{id_k} \otimes \mathrm{Tr_n}](WP_xW^*) = [\mathrm{id}_k \otimes \mathrm{Tr}_n]P_{Wx}$, the minimum output entropy of the channel Φ is

$$H^{\min}(\Phi) = \min_{x \in \mathbb{C}^d,\, \|x\|=1} H(\Phi(P_x)) = \min_{y \in \mathrm{Im} W,\, \|y\|=1} H([\mathrm{id}_k \otimes \mathrm{Tr}_n] P_y),$$

  where $V = \mathrm{Im} W \subset \mathbb{C}^k \otimes \mathbb{C}^n$ is a subspace of dimension d.
- The MOE $H^{\min}(\Phi)$ depends only on the subspace V.

Entanglement of subspaces

Eigen- and singular values

Singular value decomposition of a matrix $X \in \mathcal{M}_{k\times n}(\mathbb{C})$ ($k \le n$)

$$X = U\Sigma V^* = \sum_{i=1}^{k} \sqrt{\lambda_i(XX^*)} e_i f_i^*,$$

where e_i, f_i are orthonormal families in $\mathbb{C}^k$, $\mathbb{C}^n$, and $\lambda_1 \ge \cdots \ge \lambda_k \ge 0$ are the (squares of the) singular values of X, or the eigenvalues of XX^*.

Using the isomorphism $\mathcal{M}_{k\times n} \simeq \mathbb{C}^k \otimes \mathbb{C}^n$, X can be seen as a vector in a tensor product $x \in \mathbb{C}^k \otimes \mathbb{C}^n$. The singular value decomposition of X corresponds to the Schmidt decomposition of x

$$x = \sum_{i=1}^{k} \sqrt{\lambda_i(x)} e_i \otimes f_i.$$

The numbers $\lambda_i(x)$ are also eigenvalues of the reduced density matrix

$$XX^* = [\mathrm{id}_k \otimes \mathrm{Tr}_n] P_x = \sum_{i=1}^{k} \lambda_i(x) e_i e_i^*.$$

Entanglement of a vector

For a vector

$$x = \sum_{i=1}^{k} \sqrt{\lambda_i(x)} e_i \otimes f_i,$$

define $H(x) = H(\lambda(x)) = H(\rho) = -\sum_i \lambda_i(x) \log \lambda_i(x)$, the entropy of entanglement of the bipartite pure state x.

Note that

1. The state x is separable, $x = e \otimes f$, iff. $H(x) = 0$.
2. The state x is maximally entangled, $x = k^{-1/2} \sum_i e_i \otimes f_i$, iff. $H(x) = \log k$.

Recall that we are interested in computing

$$\begin{aligned} H^{\min}(\Phi) &= \min_{x \in \mathbb{C}^d,\, \|x\|=1} H(\Phi(P_x)) = \min_{y \in \mathrm{Im}\, W,\, \|y\|=1} H([\mathrm{id}_k \otimes \mathrm{Tr}_n] P_y) \\ &= \min_{y \in \mathrm{Im}\, W,\, \|y\|=1} H(y). \end{aligned}$$

Entanglement of a subspace

For a subspace $V \subset \mathbb{C}^k \otimes \mathbb{C}^n$, define

$$\boxed{H^{\min}(V) := \min_{y \in V,\, \|y\|=1} H(y),}$$

the minimal entanglement of vectors in V.

A subspace V is called entangled if $H^{\min}(V) > 0$, i.e. if it does not contain separable vectors $x \otimes y$.

Proposition (Parthasarathy '03)

If V is entangled, then $\dim V \leq (k-1)(n-1)$.

Example: $V_{ent} = \{x : \forall r, \sum_{i+j=r} x_{ij} = 0\}$.

Singular values of vectors from a subspace

Entropy is just a statistic, look at the set of all singular values directly !

For a subspace $V \subset \mathbb{C}^k \otimes \mathbb{C}^n$ of dimension $\dim V = d$, define the set eigen-/singular values or Schmidt coefficients

$$K_V := \{\lambda(x) : x \in V, \|x\| = 1\}.$$

$\rightsquigarrow$ Our goal is to understand K_V.

- The set K_V is a compact subset of the ordered probability simplex $\Delta_k^{\downarrow}$.
- Local invariance: $K_{(U_1 \otimes U_2)V} = K_V$, for unitary matrices $U_1 \in \mathcal{U}(k)$ and $U_2 \in \mathcal{U}(n)$.
- Monotonicity: if $V_1 \subset V_2$, then $K_{V_1} \subset K_{V_2}$.
- Recovering minimum entropies:

$$H_p^{\min}(\Phi) = H_p^{\min}(V) = \min_{\lambda \in K_V} H_p(\lambda).$$

Examples

The anti-symmetric subspace provides the (explicit) counter-example for the additivity of the p-Rényi entropy.

- Let $k = n$ and put $V = \Lambda^2(\mathbb{C}^k)$
- The subspace V is almost half of the total space: $\dim V = k(k-1)/2$.
- Example of a vector in V:

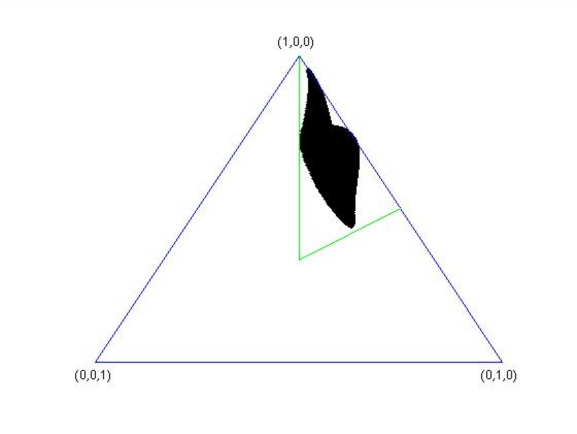
$$V \ni x = \frac{1}{\sqrt{2}}(e \otimes f - f \otimes e).$$

- Fact: singular values of vectors in V come in pairs.
- Hence, the least entropy vector in V is as above, with $e \perp f$ and $H(x) = \log 2$.
- Thus, $H^{\min}(V) = \log 2$ and one can show that

$$K_V = \{(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots) \in \Delta_k \,:\, \lambda_i \geq 0, \sum_i \lambda_i = 1/2\}.$$

Examples

$V = \mathrm{span}\{G_1, G_2\}$, where $G_{1,2}$ are 3×3 independent Ginibre random matrices.



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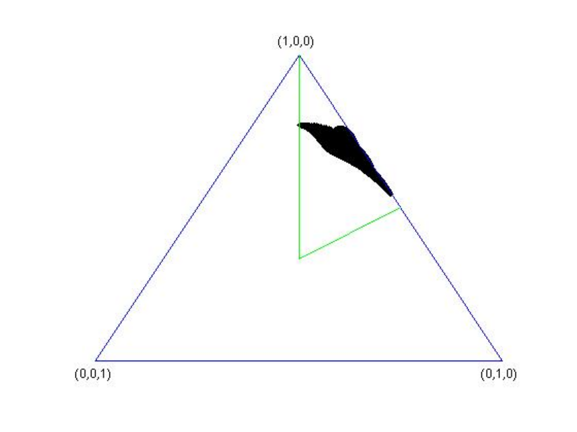
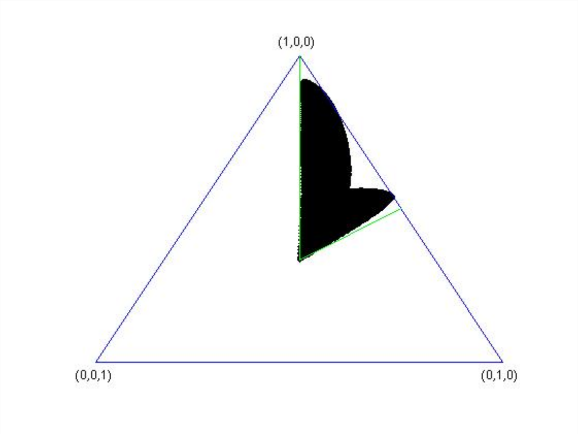


Examples

$V = \mathrm{span}\{\mathrm{I}_3, G\}$, where G is a 3×3 Ginibre random matrix.



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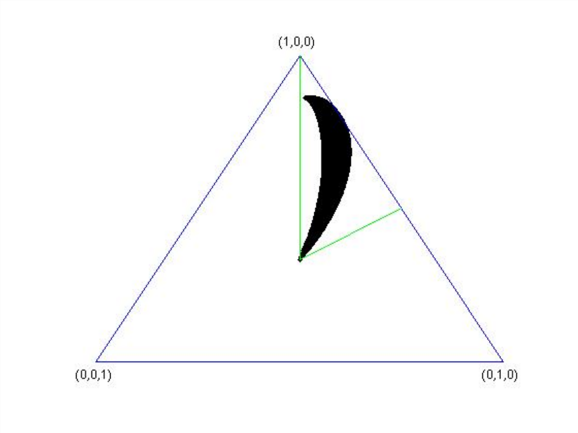


A big open problem

Find explicit examples of subspaces V with

1. large $\dim V$;
2. large $H^{\min}(V)$.

Random subspaces

Random subspaces

We are interested in random subspaces (or random channels).

- There is an uniform (or Haar) measure on the set of isometries $\{W : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n \, : \, WW^* = \mathrm{I}_d\}$: take a $kn \times kn$ Haar distributed random unitary matrix $U \in \mathcal{U}(kn)$ and take W to be the restriction of U to the first d coordinates.
- We call random quantum channels the probability distribution obtained as the push-forward of this measure through the Stinespring dilation.
- A random subspace is the image of a random isometry, $V = \mathrm{Im} W$.
- Equivalently, $V = \mathrm{span}\{U_1, \ldots, U_d\}$, where U_i are the columns of a Haar random unitary matrix $U \in \mathcal{U}(kn)$.

Main result

For the rest of the talk, we consider the following asymptotic regime: k fixed, $n \to \infty$, and $d \sim tkn$, for a fixed parameter $t \in (0,1)$.

Theorem (Belinschi, Collins, N. '10)

For a sequence of uniformly distributed random subspaces V_n, the set K_{V_n} of singular values of unit vectors from V_n converges (almost surely, in the Hausdorff distance) to a deterministic, convex subset $K_{k,t}$ of the probability simplex Δ_k

$$K_{k,t} := \{\lambda \in \Delta_k \mid \forall x \in \Delta_k, \langle \lambda, x \rangle \leq \|x\|_{(t)}\}.$$

Corollary: exact limit of the minimum output entropy

By the previous theorem, in the specific asymptotic regime t, k fixed, $n \to \infty$, $d \sim tkn$, we have the following a.s. convergence result for random quantum channels Φ (defined via random isometries $W : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n$):

$$\lim_{n \to \infty} H^{\min}(\Phi) = \min_{\lambda \in K_{k,t}} H(\lambda).$$

It is not just a bound, the exact limit value is obtained.

Theorem (Belinschi, Collins, N. '13)

The minimum entropy element of $K_{k,t}$ is of the form $(a, b, b, \ldots, b)$. The lowest dimension for which a violation of the additivity for $H^{\min}$ can be observed is $k = 183$. For large k, violations of size $1 - \varepsilon$ bits can be obtained.

Free Probability Theory

Invented by Voiculescu in the 80s to solve problems in operator algebras.

- A non-commutative probability space $(\mathcal{A}, \tau)$ is an algebra $\mathcal{A}$ with a unital state $\tau : \mathcal{A} \to \mathbb{C}$. Elements $a \in \mathcal{A}$ are called random variables.
- Examples:
  - classical probability spaces $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$;
  - group algebras $(\mathbb{C}G, \delta_e)$;
  - matrices $(\mathcal{M}_n, n^{-1}\mathrm{Tr})$;
  - random matrices $(\mathcal{M}_n(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P})), \mathbb{E} \circ n^{-1}\mathrm{Tr})$.
- Several notions of independence:
  - classical independence, implies commutativity of the random variables;
  - free independence.
- If a, b are freely independent random variables, the law of (a, b) can be computed in terms of the laws of a and b. Freeness provides an algorithm for computing joint moments in terms of marginals.
- Example: if $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are free, then

$$\tau(a_1 b_1 a_2 b_2) = \tau(a_1 a_2)\tau(b_1)\tau(b_2) + \tau(a_1)\tau(a_2)\tau(b_1 b_2) - \tau(a_1)\tau(b_1)\tau(a_2)\tau(b_2).$$

Asymptotic freeness of random matrices

Theorem (Voiculescu '91)

Let (A_n) and (B_n) be sequences of $n \times n$ matrices such that A_n and B_n converge in distribution (with respect to $n^{-1}\mathrm{Tr}$) for $n \to \infty$. Furthermore, let (U_n) be a sequence of Haar unitary $n \times n$ random matrices. Then, A_n and $U_n B_n U_n^*$ are asymptotically free for $n \to \infty$.

If A_n, B_n are matrices of size n, whose spectra converge towards μ_a, μ_b, the spectrum of $A_n + U_n B_n U_n^*$ converges to $\mu_a \boxplus \mu_b$; here, $\mu_a \boxplus \mu_b$ is the distribution of $a + b$, where $a, b \in (\mathcal{A}, \tau)$ are free random variables having distributions resp. μ_a, μ_b.

If A_n, B_n are matrices of size n such that $A_n \geq 0$, whose spectra converge towards μ_a, μ_b, the spectrum of $A_n^{1/2} U_n B_n U_n^* A_n^{1/2}$ converges to $\mu_a \boxtimes \mu_b$.

Example: truncation of random matrices

Let $P_n \in \mathcal{M}_n$ a projection of rank $n/2$; its eigenvalues are 0 and 1, with multiplicity $n/2$. Hence, the distribution of P_n converges, when $n \to \infty$, to the Bernoulli probability measure $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

Let $C_n \in \mathcal{M}_{n/2}$ be the top $n/2 \times n/2$ corner of $U_n P_n U_n^*$, with U_n a Haar random unitary matrix. What is the distribution of C_n ? Up to zero blocks, $C_n = Q_n(U_n P_n U_n^*)Q_n$, where Q_n is the diagonal orthogonal projection on the first $n/2$ coordinates of $\mathbb{C}^n$. The distribution of Q_n converges to $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

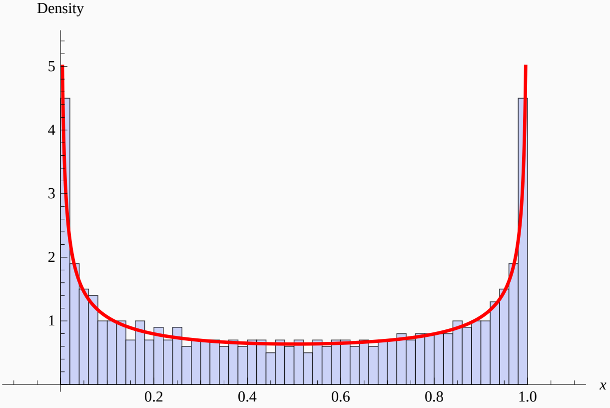
Free probability theory tells us that the distribution of C_n will converge to

$$(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1) \boxtimes (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1) = \frac{1}{\pi\sqrt{x(1-x)}}\mathbf{1}_{[0,1]}(x)dx,$$

which is the arcsine distribution.

Example: truncation of random matrices

Histogram of eigenvalues of a truncated randomly rotated projector of relative rank $1/2$ and size $n = 4000$; in red, the density of the arcsine distribution.



The t-norm

Definition

For a positive integer k, embed $\mathbb{R}^k$ as a self-adjoint real subalgebra $\mathcal{R}$ of a C^*-ncps $(\mathcal{A}, \tau)$, so that $\tau(x) = (x_1 + \cdots + x_k)/k$. Let p_t be a projection of rank $t \in (0, 1]$ in $\mathcal{A}$, free from $\mathcal{R}$. On the real vector space $\mathbb{R}^k$, we introduce the following norm, called the (t)-norm:

$$\|x\|_{(t)} := \|p_t x p_t\|_\infty,$$

where the vector $x \in \mathbb{R}^k$ is identified with its image in $\mathcal{R}$.

- One can show that $\| \cdot \|_{(t)}$ is indeed a norm, which is permutation invariant.
- When $t > 1 - 1/k$, $\| \cdot \|_{(t)} = \| \cdot \|_\infty$ on $\mathbb{R}^k$.
- $\lim_{t \to 0^+} \|x\|_{(t)} = k^{-1} |\sum_i x_i|$.

Corners of randomly rotated projections

Theorem (Collins '05)

In $\mathbb{C}^n$, choose at random according to the Haar measure two independent subspaces V_n and V_n' of respective dimensions $q_n \sim sn$ and $q_n' \sim tn$ where $s, t \in (0,1]$. Let P_n (resp. P_n') be the orthogonal projection onto V_n (resp. V_n'). Then,

$$\lim_n \|P_n P_n' P_n\|_\infty = \varphi(s,t) = \sup \operatorname{supp}((1-s)\delta_0 + s\delta_1) \boxtimes ((1-t)\delta_0 + t\delta_1),$$

with

$$\varphi(s,t) = \begin{cases} s + t - 2st + 2\sqrt{st(1-s)(1-t)} & \text{if } s + t < 1; \\ 1 & \text{if } s + t \geq 1. \end{cases}$$

Hence, we can compute

$$\|\underbrace{1, \cdots, 1}_{j \text{ times}}, \underbrace{0, \cdots, 0}_{k-j \text{ times}}\|_{(t)} = \varphi(\frac{j}{k}, t).$$

Idea of the proof

A simpler question: what is the largest maximal singular value $\max_{x \in V, \|x\|=1} \lambda_1(x)$ of vectors from the subspace V ?

$$\begin{aligned}
\max_{x \in V, \|x\|=1} \lambda_1(x) &= \max_{x \in V, \|x\|=1} \lambda_1([\mathrm{id}_k \otimes \mathrm{Tr}_n] P_x) \\
&= \max_{x \in V, \|x\|=1} \|[\mathrm{id}_k \otimes \mathrm{Tr}_n] P_x\| \\
&= \max_{x \in V, \|x\|=1} \max_{y \in \mathbb{C}^k, \|y\|=1} \mathrm{Tr}\left[([\mathrm{id}_k \otimes \mathrm{Tr}_n] P_x) \cdot P_y\right] \\
&= \max_{x \in V, \|x\|=1} \max_{y \in \mathbb{C}^k, \|y\|=1} \mathrm{Tr}\left[P_x \cdot P_y \otimes \mathrm{I}_n\right] \\
&= \max_{y \in \mathbb{C}^k, \|y\|=1} \max_{x \in V, \|x\|=1} \mathrm{Tr}\left[P_x \cdot P_y \otimes \mathrm{I}_n\right] \\
&= \max_{y \in \mathbb{C}^k, \|y\|=1} \|P_V \cdot P_y \otimes \mathrm{I}_n\|_\infty .
\end{aligned}$$

Limit of $\|P_V \cdot P_y \otimes \mathrm{I}_n\|_\infty$ for fixed y and random V ?

The set $K_{k,t}$ and t-norms

- $K_{k,t} := \{\lambda \in \Delta_k \mid \forall x \in \Delta_k, \langle \lambda, x \rangle \leq \|x\|_{(t)}\}$.
- Recall that

$$\max_{x \in V, \|x\|=1} \lambda_1(x) = \max_{y \in \mathbb{C}^k, \|y\|=1} \|P_V P_y \otimes \mathrm{I}_n\|_\infty .$$

- For fixed y, P_V and $P_y \otimes \mathrm{I}_n$ are independent projectors of relative ranks t and $1/k$ respectively.
- Thus, $\|P_V \cdot P_y \otimes \mathrm{I}_n\|_\infty \to \varphi(t, 1/k) = \|(1, 0, \ldots, 0)\|_{(t)}$.
- We can take the max over y at no cost, by considering a finite net of y's, since k is fixed.
- To get the full result $\limsup_{n\to\infty} K_{V_n} \subset K_{k,t}$, use $\langle \lambda, x \rangle$ (for all directions x) instead of λ_1.

The take-home slide

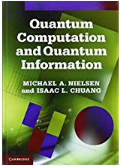
States	Deterministic	Random mixture
Classical	$x \in \{1, 2, \ldots, d\}$	$p \in \mathbb{R}^d$, $p_i \geq 0$, $\sum_i p_i = 1$
Quantum	$\psi \in \mathbb{C}^d$, $\|\psi\| = 1$	$\rho \in \mathcal{M}_d(\mathbb{C})$, $\rho \geq 0$, $\operatorname{Tr} \rho = 1$

- Random quantum states: $\rho = W / \operatorname{Tr} W$, with W a Wishart matrix.
- Used e.g. to test the power of entanglement criteria, such as the partial transposition $[\mathrm{id} \otimes \Theta](\rho_{AB})$.

Channels	Deterministic	Random mixture
Classical	$f \in \mathcal{S}_d$	Q Markov: $Q_{ij} \geq 0$ and $\forall i$, $\sum_j Q_{ij} = 1$
Quantum	$U \in \mathcal{U}(d)$	Φ CPTP map

- Random quantum channels: Stinesrping dilation $\Phi(\rho) = [\mathrm{id} \otimes \operatorname{Tr}](V \rho V^*)$ for a Haar-random isometry $V : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n$.
- Used e.g. to disprove the additivity conjecture: $H^{\min}(\Phi \otimes \Psi) = H^{\min}(\Phi) + H^{\min}(\Psi)$.

To go further - books



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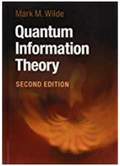

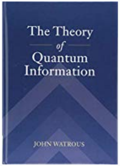

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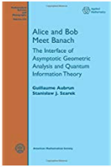

Merci!