

COMPATIBILITY OF QUANTUM MEASUREMENTS

AND

INCLUSION OF FREE SPECTRAHEDRA

joint work with Andreas BLUTHER (QMATH, Copenhagen)

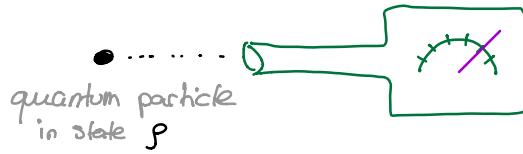
arXiv: $\begin{cases} 1807.01508 \\ 1809.04514 \end{cases}$

Goal: Relate $\left. \begin{array}{l} \text{Compatibility of POMs in QM} \\ \text{Noise robustness} \end{array} \right\} \equiv \left. \begin{array}{l} \text{Inclusion of the matrix diamond} \\ \text{Inclusion constants for---} \end{array} \right\}$

① Compatibility of quantum measurements

→ quantum states ≡ density matrices $\{ \rho \in M_d(\mathbb{C}) : \rho \geq 0 \text{ and } \text{Tr } \rho = 1 \}$
└ PSD order

→ measurement results in QM are random



measurement apparatus produces an outcome
 $i \in \{1, 2, \dots, k\}$
with probabilities (p_1, \dots, p_k)
 $p_i \geq 0, \sum p_i = 1$

Axioms of QM: the map $\rho \mapsto p = (p_1, \dots, p_k)$ must be:

- linear
 - positive $p = (p_1, \dots, p_k) \in \mathbb{R}_+^k$
 - normalized $\sum p_i = 1$
- ⇒ $p_i = \text{Tr}(A_i \rho)$ for $A_i \in M_d(\mathbb{C})$
s.t. $A_i \geq 0$ and $\sum A_i = I_d$

Def A k -valued **POM** is a k -tuple $A = (A_1, \dots, A_k) \in M_d(\mathbb{C})^k$ s.t. $A_i \geq 0$ and $\sum_{i=1}^k A_i = I_d$

Example $k=2$ $A = (E, I-E)$ for some $0 \leq E \leq I$. E is called a **quantum effect**.

$$([0, 0], [0, 0]) ; \left(\frac{1}{2} [1, 1], \frac{1}{2} [-1, -1] \right) ; \left(\frac{1}{3} I, \frac{2}{3} I \right)$$

Compatibility Can two POMs A, B be measured "at the same time"?

Def Two POMs A, B are said to be **compatible** if there exists a POM C called a **joint POM** such that w.r.t. quantum state ρ , the probabilities $p = [\text{Tr}(A_i \rho)]_{i=1}^k$ and $q = [\text{Tr}(B_j \rho)]_{j=1}^l$ can be obtained as **classical post-processing** of $r = [\text{Tr}(C_x \rho)]$

Fact A, B compatible if $\exists C_{ij} \geq 0, i \in [k], j \in [l]$ s.t. $\begin{cases} \forall i \sum_j C_{ij} = A_i \\ \forall j \sum_i C_{ij} = B_j \end{cases}$

Two effects E, F are compatible if $\exists X_{11}, X_{12}, X_{21}, X_{22} \geq 0$ s.t. $\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = E$ $\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = I - E$
 $\begin{matrix} " \\ F \end{matrix} \quad \begin{matrix} " \\ I - F \end{matrix}$

This can be generalized to g -tuples of POVMs: $A^{(1)}, A^{(2)}, \dots, A^{(g)}$ are compatible if

$$\exists B = (B_{i_1 \dots i_g}) \text{ s.t. } \forall n \in [g], \forall x \in [k_n], A_x^{(n)} = \sum_{i_1, \dots, i_{n-1}, i_{n+1}, \dots, i_g} B_{i_1 \dots i_n x i_{n+1} \dots i_g}$$

Examples

① $A = (a_1 I, \dots, a_k I)$ and $B = (b_1 I, \dots, b_k I)$ are compatible: $C_{ij} := a_i b_j I$
 \uparrow these are called trivial POVMs

② $(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix})$ and $(\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix})$ are not compatible

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow X_{11} = 0, \text{ same for all the others} \quad \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

③ If $[A_i, B_j] = 0 \ \forall i, j \Rightarrow A$ and B are compatible

$$\text{take } C_{ij} := A_i B_j = A_i^{1/2} B_j A_i^{1/2} \geq 0.$$

Fact POVMs can be made compatible by adding noise, i.e. mixing trivial POVMs

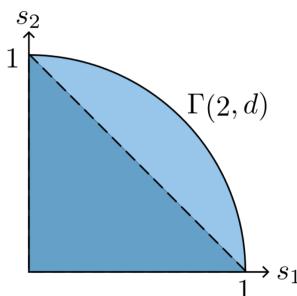
$$(E, I - E) \mapsto s(E, I - E) + (1-s)\left(\frac{I}{2}, \frac{I}{2}\right) \quad \text{or} \quad E \mapsto sE + (1-s)\frac{I}{2}$$

Def The compatibility region for g effects in Γ_d is the set

$$\Gamma(g, d) := \{s \in [0, 1]^g : \forall \text{ effects } 0 \leq E_1, \dots, E_g \leq I_d, s_i E_i + (1-s_i) \frac{I_d}{2} \text{ are comp.}\}$$

- Facts
- $(0, \dots, 0, 1, 0, \dots, 0) \in \Gamma(g, d)$: measure $(E_m, I_d - E_n)$, flip a coin for the others
 \uparrow position n
 - $\Gamma(g, d)$ is convex. In particular $(\frac{1}{g}, \dots, \frac{1}{g}) \in \Gamma(g, d)$

- $g=2$



$\Gamma(2, d)$ is a quarter-circle $\forall d$

② Free spectrahedra

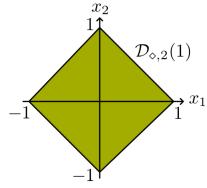
Def Let $A \in (M_d(\mathbb{C}))^g$. The **free spectrahedron at level n** is the set

$$\mathcal{D}_A(n) := \{ X \in (M_n(\mathbb{C}))^g : \sum_{i=1}^g A_i \otimes X_i \leq I_{dn} \}$$

The **free spectrahedron** is the union of these levels: $\mathcal{D}_A := \bigsqcup_{n \in \mathbb{N}} \mathcal{D}_A(n)$.

Example The **matrix diamond** $\mathcal{D}_{\diamond,g}(n) := \{ X : \sum \varepsilon_i X_i \leq I_n \text{ } \forall \varepsilon \in \{\pm 1\}^g \}$

for $g=2$:



This can be obtained

$$\text{from } A_1 = \text{diag}(1, 1, -1, -1)$$

$$A_2 = \text{diag}(1, -1, 1, -1)$$

Def An **operator system** is a linear subspace $\mathcal{L} \subseteq M_d(\mathbb{C})$ s.t. $I \in \mathcal{L}$ and $\mathcal{L} = \mathcal{L}^*$.

A linear map $\Phi : \mathcal{L} \rightarrow M_d(\mathbb{C})$ is called **n -positive** if $\Phi \otimes \text{id}_n : \mathcal{L} \otimes M_n(\mathbb{C}) \rightarrow M_d(\mathbb{C}) \otimes M_n(\mathbb{C})$ is a positive map. Φ is called **completely positive** if it is n -positive $\forall n \in \mathbb{N}$.

Theorem [Helton, Klep, McCullough '13] Let $A \in (M_d(\mathbb{C}))^g$, $B \in (M_d(\mathbb{C}))^g$ s.t. $\mathcal{D}_A(1)$ is bounded.

The unital linear map $\Phi : \text{span}\{I, A_1, \dots, A_g\} \rightarrow M_d(\mathbb{C})$ given by $\Phi(A_i) = B_i \forall i \in [g]$

is n -positive iff $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$. In particular, Φ is CP iff $\mathcal{D}_A \subseteq \mathcal{D}_B$.

\Rightarrow Obviously $\mathcal{D}_A \subseteq \mathcal{D}_B \Rightarrow \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$. How about the reverse inclusion?

Def The **inclusion set** for the matrix diamond is

$$\Delta(g, d) = \{ s \in [0, 1]^g : \forall B \in (M_d(\mathbb{C}))^g, \mathcal{D}_{\diamond,g}(1) \subseteq \mathcal{D}_B(1) \Rightarrow s \cdot \mathcal{D}_{\diamond,g} \subseteq \mathcal{D}_B \}$$

\uparrow
 $\{ (s_1 x_1, \dots, s_g x_g) : \dots \}$

Facts about $\Delta(g, d)$

- [Helton, Klep, McCullough, Schweighofer '19]: $\frac{1}{2d}(1, \dots, 1) \in \Delta(g, d)$

- [Passer, Shalit, Solel '18]: $QC_g := \{ s \in [0, 1]^g : \sum s_i^2 \leq 1 \} \subseteq \Delta(g, d)$

③ Connecting the two problems

Theorem [Bluhm-N '19] Let $E \in (\mathbb{M}_d(\mathbb{C}))^g$ be a g -tuple of self-adjoint matrices. Then:

1. $\mathcal{D}_{\Delta, g}(1) \subseteq \mathcal{D}_{2E-I}(1)$ iff E_1, \dots, E_g are quantum effects.
2. $\mathcal{D}_{\Delta, g} \subseteq \mathcal{D}_{2E-I}$ iff E_1, \dots, E_g are compatible.
3. $\mathcal{D}_{\Delta, g}(n) \subseteq \mathcal{D}_{2E-I}(n)$ for $n \in [d]$ iff \forall isometry $V: \mathbb{C}^m \rightarrow \mathbb{C}^d$, the compressions $V^* E_i V, \dots, V^* E_g V$ are compatible quantum effects.

We also have $\Gamma(g, d) = \Delta(g, d)$.

Proof ideas level 1: consider the extreme points $\pm e_i$ of the matrix diamond.

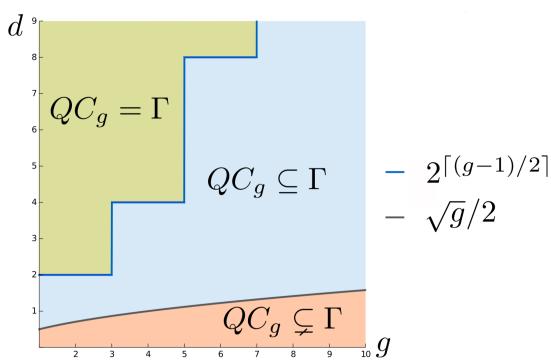
- Free inclusion holds iff the unital map

$$\Phi: I_2^{\otimes(i-1)} \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \otimes I_2^{\otimes(g-i)} \mapsto 2E_i - I_d \text{ is completely positive}$$

- Arveson's extension theorem: Φ has a positive extension $\tilde{\Phi}$ to \mathbb{C}^{2^g} .

- Basis x_η of \mathbb{C}^{2^g} : $X_\eta := \tilde{\Phi}(x_\eta)$ is a joint POU for E_1, \dots, E_g iff $\tilde{\Phi}$ positive.

What is known about $\Gamma(g, d) = \Delta(g, d)$



- $QC_g = \{ s \in [0, 1]^g : \sum_{i=1}^g s_i^2 \leq 1 \}$
- "top region": lower and upper bounds from [PSS'18] coincide
- "bottom region": cannot have equality because of $\frac{1}{2d}(1, \dots, 1) \in \Gamma(g, d)$
- "central region": open

Maximally incompatible effects [PSS'18]: use families of anti-commuting, s.a. unitary op.

For $d = 2^k$, there exist $2k+1$ such operators in $\mathbb{M}_d(\mathbb{C})$ [Newman '32]

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$F_1^{(1)} = \sigma_x; \quad F_2^{(1)} = \sigma_y; \quad F_3^{(1)} = \sigma_z;$$

$$F_i^{(k+1)} = \sigma_x \otimes F_i^{(k)} \quad \forall i \in [2k+1]; \quad F_{2k+2}^{(k+1)} = \sigma_y \otimes I_{2^k}; \quad F_{2k+3}^{(k+1)} = \sigma_z \otimes I_{2^k}$$

④ Generalizations

→ we have discussed compatibility of quantum effects, i.e. 2-outcome POVMs ($E, I-E$)

→ in the general case, we have POVMs $A^{(n)}$ with k_n outcomes.

→ the Theorem holds, with the matrix diamond replaced by the **matrix jewel**

- at level 1: $\mathcal{D}_{J, (k_1, \dots, k_g)}^{(1)} := \mathcal{D}_{J, k_1}^{(1)} \oplus \dots \oplus \mathcal{D}_{J, k_g}^{(1)}$
where $\mathcal{D}_{J, k_l}^{(1)}$ is isomorphic to a $(l-1)$ -simplex.

and " \oplus " is the **direct sum** of convex sets:

$$K_1 \oplus K_2 := \text{conv}(\{(x, 0) : x \in K_1\} \cup \{(0, y) : y \in K_2\})$$

not to be confused with $K_1 \times K_2 = \{(x, y) : x \in K_1, y \in K_2\}$

- extend this to the free level: $\mathcal{D}_{J, \mathbb{N}} := \mathcal{D}_{J, k_1} \widehat{\oplus} \dots \widehat{\oplus} \mathcal{D}_{J, k_g}$

→ the matrix jewel has fewer symmetries than the diamond. Many "classical" results
do not apply

→ we have very loose bounds, based mainly on **symmetrization arguments** or on
QIT techniques such as { **quantum cloning** (for LB)
incompatibility criteria (for UB) }

→ the Theorem also holds in very different settings, such as

work in progress {

- compatibility of **quantum channels**
- **GPTs** (generalized probabilistic theories)
↳ min- and max-tensor products, tensor norms.