Quantum information theory and Reznick's Positivstellensatz

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Talk outline

Sums of squares and Reznick's Positivstellensatz

Poylnomials vs. symmetric operators

The complex Positivstellensatz

Sums of squares and Reznick's Positivstellensatz

Hilbert's 17th problem

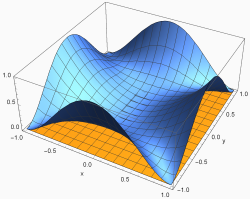
- $\mathbb{R}[x] \ni P(x) \geq 0 \iff P = Q_1(x)^2 + Q_2(x)^2$, for $Q_{1,2} \in \mathbb{R}[x]$
- $\mathrm{Pos}(d, n) := \{P \in \mathbb{R}[x_1, \ldots, x_d]$ hom. of deg. $2n$, $P(x) \geq 0, \forall x\}$
- $\mathrm{SOS}(d, n) := \{\sum_i Q_i^2$ with $Q_i \in \mathbb{R}[x_1, \ldots, x_d]$ hom. of deg. $n\}$
- Hilbert 1888:

$$\mathrm{SOS}(d, n) \subseteq \mathrm{Pos}(d, n), \text{ eq. iff } (d, n) \in \{(d, 1), (2, n), (3, 2)\}$$

- The Motzkin polynomial $x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2$ is positive but not SOS
- Membership in SOS can be efficiently decided with a semidefinite program (SDP) and provides an algebraic certificate for positivity

More on the Motzkin polynomial

The non-homogeneous Motzkin polynomial (set $z = 1$) $x^4y^2 + y^4 + x^2 - 3x^2y^2$ can be seen to be positive by the AMGM inequality



There exist computer algebra packages to check SOS and perform polynomial optimization using SOS ([NC]SOSTOOLS, Gloptipoly)

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>> syms x y z; findsos(x^4*y^2 + y^4 + x^2 -
3*x^2*y^2)

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No sum of squares decomposition is found.
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Reznick's Positivstellensatz

- Hilbert 1900, Artin 1927:

$$P \geq 0 \iff P = \sum_i \frac{Q_i^2}{R_i^2}$$

In particular, if $P \geq 0$, there exists R such that $R^2 P$ is SOS

Theorem. [Reznick 1995]

Let $P \in \mathrm{Pos}(d, k)$ such that $m(P) := \min_{\|x\|=1} P(x) > 0$. Let also $M(P) := \max_{\|x\|=1} P(x)$. Then, for all

$$n \geq \frac{dk(2k-1)}{2 \ln 2} \frac{M(P)}{m(P)} - \frac{d}{2},$$

we have

$$(x_1^2 + \cdots + x_d^2)^{n-k} P(x) = \sum_{j=1}^{r} (a_1^{(j)} x_1 + \cdots a_d^{(j)} x_d)^{2n}.$$

In particular, $\|x\|^{2(n-k)} P$ is SOS.

Poylnomials vs. symmetric operators

From the symmetric subspace to polynomials — $\mathbb{R}$

- Homogeneous polynomials of degree n in d real variables $x_1, \ldots, x_d$ are in one-to-one correspondence with symmetric tensors:

$$\vee^n \mathbb{R}^d \ni v \rightsquigarrow P_v(x_1, \ldots, x_d) = \langle x^{\otimes n}, v \rangle$$

where $x = (x_1, \ldots, x_d)$ is the vector of variables

- $n = 1,\ P_v(x) = \sum_{i=1}^d v_i x_i$
- $|GHZ\rangle = |000\rangle + |111\rangle \rightsquigarrow P_{|GHZ\rangle}(x, y) = x^3 + y^3$
- $|W\rangle = |001\rangle + |010\rangle + |001\rangle \rightsquigarrow P_{|W\rangle}(x, y) = 3x^2 y$
- If $|\Omega\rangle = \sum_{i=1}^d |ii\rangle$, then $P_{|\Omega\rangle^{\otimes n}}(x_1, \ldots, x_d) = (\sum_{i=1}^d x_i^2)^n = \|x\|^{2n}$

- We denote $d[n] := \dim \vee^n \mathbb{R}^d = \binom{n+d-1}{n}$

From the symmetric subspace to polynomials — $\mathbb{C}$

- In the complex case, we are interested in bi-homogeneous polynomials of degree n in d complex variables: $P(z_1, \ldots, z_d)$ is hom. in the variables z_i and also in $\bar{z}_i$.
- Bi-hom. polynomials are in one-to-one correspondence with operators on $\vee^n \mathbb{C}^d$:

$$P(z_1, \ldots, z_d) = \langle z^{\otimes n} | W | z^{\otimes n} \rangle$$

- Self-adjoint W are associated to real, bi-hom. polynomials
- $\|z\|^{2n} = \langle z^{\otimes n} | P_{sym}^{(d,n)} | z^{\otimes n} \rangle$
- More generally, polynomials which are bi-hom. of degree n in complex variables $z_1, \ldots, z_d$ and, separately, bi-hom. of degree k in complex variables $u_1, \ldots, u_D$ are in one-to-one correspondence with operators on $\vee^n \mathbb{C}^d \otimes \vee^k \mathbb{C}^D$:

$$Q(z_1, \ldots, z_d, u_1, \ldots, u_D) = \langle z^{\otimes n} \otimes u^{\otimes k} | W | z^{\otimes n} \otimes u^{\otimes k} \rangle$$

The different notions of positivity

A self-adjoint matrix $W \in \mathcal{B}(\vee^n \mathbb{C}^d)$ is called

- block-positive if $\langle z^{\otimes n} | W | z^{\otimes n} \rangle \geq 0$, $\forall z \in \mathbb{C}^d$
- positive semidefinite (PSD) if $\langle u | W | u \rangle \geq 0$, $\forall u \in \vee^n \mathbb{C}^d$
- separable if $W \in \mathrm{conv}\{|z\rangle\langle z|^{\otimes n}\}_{z \in \mathbb{C}^d}$

We have: W separable $\implies$ W PSD $\implies$ W block-positive

- W is block-positive $\iff$ P_W is non-negative:

$$P_W(z) = \langle z^{\otimes n} | W | z^{\otimes n} \rangle \geq 0, \qquad \forall z \in \mathbb{C}^d$$

- W is PSD $\iff$ P_W is Sum Of hom. Squares:

$$W = \sum_j \lambda_j |w_j\rangle\langle w_j| \implies P_W(z) = \sum_j \lambda_j |\langle z^{\otimes n}, w_j \rangle|^2$$

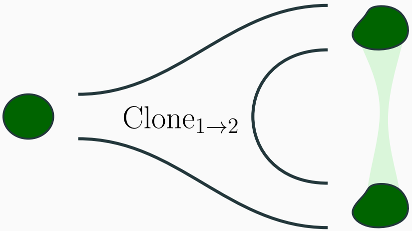
- W is separable $\iff$ P_W is Sum Of hom. Powers:

$$W = \sum_j t_j |a_j\rangle\langle a_j|^{\otimes n} \implies P_W(z) = \sum_j t_j |\langle z, a_j \rangle|^{2n}$$

Tensoring with the identity

- For $k \le n$, let $\mathrm{Tr}^*_{k\to n} : \mathcal{B}(\vee^k \mathbb{C}^d) \to \mathcal{B}(\vee^n \mathbb{C}^d)$ be the map
$$\mathrm{Tr}^*_{k\to n}(W) = P^{(d,n)}_{sym} \left[W \otimes I_d^{\otimes(n-k)} \right] P^{(d,n)}_{sym}$$
- We have: $P_{\mathrm{Tr}^*_{k\to n}(W)}(z) = \|z\|^{2(n-k)} P_W(z)$

$\mathrm{Clone}_{k\to n} := \frac{d[k]}{d[n]} \mathrm{Tr}^*_{k\to n}$ is the optimal Keyl-Werner cloning quantum channel: among all quantum channels sending states $\rho^{\otimes k}$ to symmetric n-partite states σ, it is the one which achieves the largest fidelity between ρ and $\mathrm{Tr}_{2\cdots n}\,\sigma$



The partial trace

For $k \le n$, let $\mathrm{Tr}_{n\to k} : \mathcal{B}(\vee^n \mathbb{C}^d) \to \mathcal{B}(\vee^k \mathbb{C}^d)$ be the partial trace

$$\mathrm{Tr}_{n\to k}(W) = \left[\mathrm{id}^{\otimes k} \otimes \mathrm{Tr}^{\otimes(n-k)}\right](W)$$

Lemma.

We have: $P_{\mathrm{Tr}_{n\to k}(W)} = ((n)_{n-k})^{-2} \Delta_{\mathbb{C}}^{n-k} P_W$, where $(x)_p = x(x-1)\cdots(x-p+1)$ and $\Delta_{\mathbb{C}}$ is the complex Laplacian

$$\Delta_{\mathbb{C}} = \sum_{i=1}^{d} \frac{\partial^2}{\partial \bar{z}_i \partial z_i}$$

Lemma (complex Bernstein inequality). ← we need analysis here

For any $W = W^* \in \mathcal{B}(\vee^n \mathbb{C}^d)$ we have

$$\forall \|z\| \le 1, \qquad \left|(\Delta_{\mathbb{C}}^s P_W)(z)\right| \le 4^{-s}(2d)^s (2n)_{2s} M(W)$$

The Dictionary	
Sym. operators $\in \mathcal{B}(\vee^n \mathbb{C}^d)$	Polynomials (d vars, bi-hom. deg. n)
W	$P_W(z) = \langle z^{\otimes n} \| W \| z^{\otimes n} \rangle$
Positivity notions	
block-positive	non-negative
positive semidefinite	Sum Of Squares
separable	Sum Of Powers
Operations	
Tensor with identity	mult. with the norm^2
Partial trace	complex Laplacian

The complex Positivstellensatz

A complex version of Reznick's PSS

Theorem.

Consider $W = W^* \in \mathcal{B}(\vee^k \mathbb{C}^d \otimes \mathbb{C}^D)$ with $m(W) > 0$ and $k \geq 1$. Then, for any

$$n \geq \frac{dk(2k-1)}{\ln\left(1 + \frac{m(W)}{M(W)}\right)} - k$$

with $n \geq k$, we have

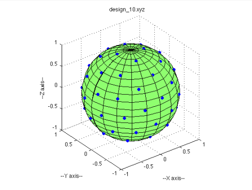
$$\|x\|^{2(n-k)} P_W(x,y) = \int P_{\tilde{W}}(\varphi, y) |\langle \varphi, x \rangle|^{2n} \mathrm{d}\varphi$$

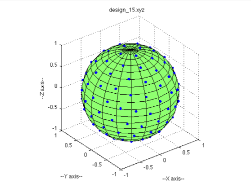
with $P_{\tilde{W}}(\varphi, y) \geq 0$ for all $\varphi \in \mathbb{C}^d$ and $y \in \mathbb{C}^D$, where $\tilde{W} \in \mathcal{B}(\vee^k \mathbb{C}^d \otimes \mathbb{C}^D)$ is explicitly computable in terms of W, and $\mathrm{d}\varphi$ is any $(n+k)$-spherical design. In the case $k = 1$, the bound on n can be improved to $n \geq dM(W)/m(W) - 1$.

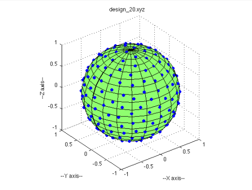
A similar result was obtained by To and Yeung with worse bounds and in a less general setting, by "complexifying" Reznick's proof

Spherical designs

- A complex n-spherical design in dimension d is a probability measure $\mathrm{d}\varphi$ on the unit sphere of $\mathbb{C}^d$ which approximates the uniform measure $\mathrm{d}z$ in the following sense: for any degree n bi-hom. polynomial $P(z)$ in d complex variables, $\int P(\varphi)\mathrm{d}\varphi = \int P(z)\mathrm{d}z$
- Equivalently, $\int |\varphi\rangle\langle\varphi|^{\otimes n}\mathrm{d}\varphi = P_{sym}^{(d,n)}/d[n]$
- For all d, n, there exist finite n-designs: the measure $\mathrm{d}\varphi$ has support of size $\leq (n+1)^{2d}$; in particular, the integral in the main theorem can be a finite sum
- Designs of orders 60, 120, 216 in $\mathbb{R}^3$ ©John Burkardt







Proof idea

$$\|x\|^{2(n-k)} P_W(x,y) = \int P_{\tilde{W}}(\varphi, y) |\langle \varphi, x \rangle|^{2n} \mathrm{d}\varphi$$

- We want to transform a non-negative polynomial into a sum of powers by multiplying with some power of the norm
- In terms of operators, this amounts to transforming a block-positive operator into a separable operator
- Ansatz: use the measure-and-prepare map

$$\mathrm{MP}_{n \to k} : \mathcal{B}(\vee^n \mathbb{C}^d) \to \mathcal{B}(\vee^k \mathbb{C}^d)$$
$$X \mapsto d[n] \int \langle \varphi^{\otimes n} | X | \varphi^{\otimes n} \rangle |\varphi\rangle\langle\varphi|^{\otimes k} \mathrm{d}\varphi,$$

  for some $(n+k)$-spherical design $\mathrm{d}\varphi$
- The linear map $\mathrm{MP}_{n \to k}$ is completely positive, and it is normalized to be trace preserving (i.e. it is a quantum channel)

Chiribella's identity

Theorem [Chiribella 2010].

For any $k \leq n$, we have

$$\mathrm{MP}_{n \to k} = \sum_{s=0}^{k} c(n,k,s)\, \mathrm{Clone}_{s \to k} \circ \mathrm{Tr}_{n \to s},$$

where $c(n,k,s) = \binom{n}{s}\binom{k+d-1}{k-s} / \binom{n+k+d-1}{k}$.

- $c(n,k,\cdot)$ is a probability distribution: $\sum_{s=0}^{k} c(n,k,s) = 1$
- The proof is a straightforward computation in the group algebra of $G = \mathcal{S}_{n+k}$:

$$\varepsilon_G = \sum_{s=0}^{\min(n,k)} \frac{\binom{n}{s}\binom{k}{s}}{\binom{n+k}{n}} \varepsilon_H \sigma_s \varepsilon_H$$

  where ε_X is the average of the elements in X, $H = \mathcal{S}_n \times \mathcal{S}_k \leq G$ is a Young subgroup and σ_s is some permutation swapping s elements from $[1,n]$ with s elements from $[n+1, n+k]$

The result is about the interplay between Clone and MP

- The equality $\|x\|^{2(n-k)} P_W(x,y) = \int P_{\tilde{W}}(\varphi, y) |\langle \varphi, x\rangle|^{2n} \mathrm{d}\varphi$ reads, in terms of linear maps over symmetric spaces

$$\mathsf{Clone}_{k\to n} \otimes \mathsf{id}_D = [\mathsf{MP}_{k\to n} \circ \Psi] \otimes \mathsf{id}_D$$

- The fact that the polynomial $P_{\tilde{W}}$ is non-negative reads

$$\tilde{W} := \Psi(W) \text{ is block-positive } \iff \langle z^{\otimes n} | \tilde{W} | z^{\otimes n} \rangle \geq 0$$

- Re-write the Chiribella identity as

$$\begin{aligned} \mathsf{MP}_{n\to k} &= \sum_{s=0}^{k} c(n,k,s)\, \mathsf{Clone}_{s\to k} \circ \mathsf{Tr}_{n\to s} \\ &= \sum_{s=0}^{k} c(n,k,s)\, \mathsf{Clone}_{s\to k} \circ \mathsf{Tr}_{k\to s} \circ \mathsf{Tr}_{n\to k} \\ &= \Phi^{(n)}_{k\to k} \circ \mathsf{Tr}_{n\to k} \end{aligned}$$

Invert the Chiribella formula

- $\mathrm{MP}_{n\to k} = \Phi^{(n)}_{k\to k} \circ \mathrm{Tr}_{n\to k}$

Key fact.

The linear map $\Phi^{(n)}_{k\to k} : \vee^k \mathbb{C}^d \to \vee^k \mathbb{C}^d$ is invertible, with inverse

$$\Psi^{(n)}_{k\to k} := \sum_{s=0}^{k} q(n,k,s)\,\mathrm{Clone}_{s\to k} \circ \mathrm{Tr}_{k\to s}$$

with

$$q(n,k,s) := (-1)^{s+k} \frac{\binom{n+s}{s}\binom{k}{s}}{\binom{n}{k}} \frac{d[k]}{d[s]}$$

- Hence, up to some constants, $\mathrm{Clone}_{k\to n} = \mathrm{MP}_{k\to n} \circ \Psi^{(n)}_{k\to k}$
- Final step: use hypotheses on $n, k, m(W), M(W)$ to ensure $\Psi^{(n)}_{k\to k}(W)$ is block-positive whenever W is (strictly) block-positive

Use the Bernstein inequality to prove $P_{\tilde{W}}$ non-negative

- Assume, wlog, $D = 1$, i.e. there is no y

$$\begin{aligned} P_{\tilde{W}}(\varphi) &= \sum_{s=0}^{k} q(n,k,s)\langle\varphi^{\otimes k}|\,\mathrm{Clone}_{s\to k}\circ \mathrm{Tr}_{k\to s}(W)|\varphi^{\otimes k}\rangle \\ &= \sum_{s=0}^{k} q(n,k,s)\|\varphi\|^{2(k-s)}\langle\varphi^{\otimes s}|\,\mathrm{Tr}_{k\to s}(W)|\varphi^{\otimes s}\rangle \\ &= \sum_{s=0}^{k} q(n,k,s)\|\varphi\|^{2(k-s)} P_{\mathrm{Tr}_{k\to s}(W)}(\varphi) \\ &= \sum_{s=0}^{k} \hat{q}(n,k,s)\|\varphi\|^{2(k-s)}(\Delta_{\mathbb{C}}^{k-s} p_W)(\varphi) \end{aligned}$$

- Use the complex version of the Bernstein inequality to ensure that

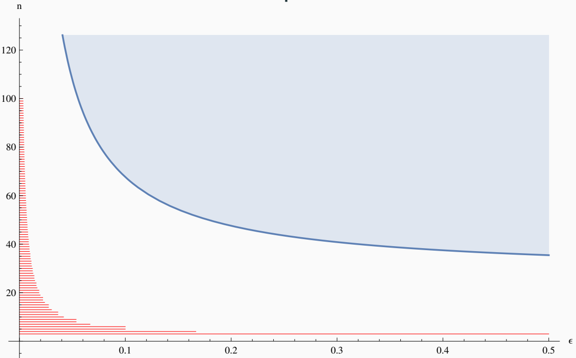
$$P_{\tilde{W}}(\varphi) \geq \left[m(W)\tilde{q}(n,k,k) - M(W)\sum_{s=0}^{k-1}|\tilde{q}(n,k,s)| \right] \geq 0$$

How good are the bounds?

- Consider the modified Motzkin polynomial

$$P_\varepsilon(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2 + \varepsilon(x^2+y^2+z^2)$$

- We have $m(P_\varepsilon) = \varepsilon$; $M(P_\varepsilon) = \varepsilon + 4/27$
- Let $P_{n,\varepsilon}(x,y,z) := (x^2+y^2+z^2)^{n-3}P_\varepsilon(x,y,z)$. If a PSS decomposition holds, then the $[2p,2q,2r]$ coefficient of $P_{n,\varepsilon}$ must be positive $\rightsquigarrow$ lower bound on optimal n



Thank you!

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