LIEB-ROBINSON BOUNDS

References → lecture notes of Ueltschi (sec 4)

→ Nachtergaele, Sims - LR bounds in QMB physics, arXiv 1004.2086

Setup • $\Lambda \subseteq \mathbb{Z}^D$ finite subset $\rightsquigarrow$ Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathbb{C}^d$; $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$

- local observables $X \subseteq \Lambda$ $\mathcal{A}_X = \{ A \otimes I_{\Lambda \setminus X} \}$
- Total hamiltonian $H_\Lambda = \sum_{X \subseteq \Lambda} h_X$

  Example $h_X = 0$ unless $X =$ ●—● or ┃

- dynamics (no thermodynamic limit here)

$$\alpha_t^\Lambda (A) = e^{itH_\Lambda} A \, e^{-itH_\Lambda} =: A_t$$

- $d(x,y)$ graph distance between $x, y \in \Lambda$
- $d(X,Y) = \min_{\substack{x \in X \\ y \in Y}} d(x,y)$
- $\mathrm{diam}(X) = \max_{x,y \in X} d(x,y)$
- $|X|$ = cardinality of X

Definition For a set of local hamiltonians $\underline{h} = \{h_X\}_{X \subseteq \Lambda}$ and $c > 0$

$$\| \underline{h} \|_c := \sup_{x \in \Lambda} \sum_{X \ni x} \| h_X \| \cdot |X| \cdot e^{c\,\mathrm{diam}(X)}$$

Example $D = 2$, $h_X = 0$ unless $X =$ ●—● or ┃ : $\| \underline{h} \|_c = 4 \cdot \|h\| \cdot 2 \cdot e^{c \cdot 1} = 8 e^c \|h\|$

Theorem [L-R bound] $\Lambda \subseteq \mathbb{Z}^D$ finite, A supported on X, B on Y, $d(X,Y) > 0$ (i.e. $X \cap Y = \emptyset$)

Then, $\forall t \in \mathbb{R}$, $\forall c > 0$ $\| [A_t, B] \| \leq 2 \|A\| \, \|B\| \, |X| \, e^{-c\,d(X,Y)} \left(e^{2 \|\underline{h}\|_c |t|} - 1 \right)$

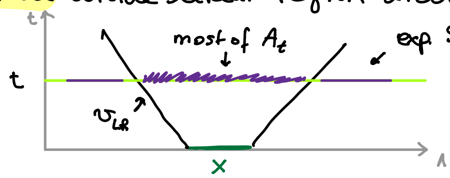
Remark speed of "information propagation"

$$e^{-c\left[d(X,Y) - \underbrace{\frac{2\|\underline{h}\|_c}{c}}_{v_{LR}} |t| \right]} \quad \rightarrow \quad v_{LR} = \frac{2 \|\underline{h}\|_c}{c}$$

Example $v_{LR,\,opt} = \min_{c>0} \frac{2 \cdot 8\, e^{c} \|h\|}{c} = 16\, e \|h\|$

Remark $\operatorname{supp} A_t = \alpha_t^{\Lambda}(A) = \Lambda$ for any $t>0$ (non-trivial h)

but: exp. small outside linear region around $X = \operatorname{supp}(A = A_0)$



Proof elements

Lemma Let $(A_t)_{t\in\mathbb{R}}$ be a family of norm-preserving oper. acting on a Banach space.

Consider ODEs: $\frac{d}{dt} X_t = A_t(X_t)$

$\frac{d}{dt} Y_t = A_t(Y_t) + B_t$.

Then, $\| Y_t - X_t \| \le \int_0^t \|B_s\|\, ds$

Proof step 1 Compute $\frac{d}{dt}[A_t, B]$ and UB $\|[A_t, B]\|$ "locally"

$\rightarrow$ $\frac{d}{dt}[A_t, B] = \frac{d}{dt}[e^{itH} A\, e^{itH}, B] = [\frac{d}{dt} e^{itH} A\, e^{-itH}, B] = [i[H, A_t], B]$

$\rightarrow$ use Jacobi: $[[x,y],z] = [x,[y,z]] - [y,[x,z]]$

$\rightarrow$ $\frac{d}{dt}[A_t, B] = i[[H, A_t], B] = i[H, [A_t, B]] - i[A_t, [H, B]]$

Important remark $[H, A_0] = \sum_z [h_z, A] = \sum_{z\cap X \neq \emptyset} [h_z, A]$ $\rightsquigarrow$ locality

$\rightarrow$ let $H' := \sum_{z\cap X\neq\emptyset} h_z$

$\rightarrow$ $[H, A_t] = [H_{-t}, A]_t = e^{itH}[e^{-itH} H e^{itH}, A] e^{-itH} = [H, A]_t = [H', A]_t = [H'_t, A_t]$

$\rightarrow$ so $\frac{d}{dt}[A_t, B] = \underbrace{i[H'_t, [A_t, B]]}_{\text{norm-preserving operator acting on } [A_t, B]} - i[A_t, [H'_t, B]]$ $\rightsquigarrow$ use Lemma

→ we get $\|[A_t,B]\| \le \|[A,B]\| + \int_0^{|t|} \|[A_s,[H'_s,B]]\|\,ds$

$$\le \overset{X\cap Y=\emptyset}{0} + 2\|A\|\int_0^{|t|}\|[H'_s,B]\|\,ds \quad \text{with } H'=\sum_{Z\cap X\neq\emptyset} h_Z$$

Proof Step 2 Evaluate recursively $\int_0^{|t|}\|[H'_s,B]\|\,ds$

→ define $g_B(Z,t) := \sup_{\substack{M\in\mathcal{A}_Z\\ M\neq 0}} \frac{\|[M_t,B]\|}{\|M\|}$. want UB on $g(X,t)$

→ $t=0$: $g_B(Z,0)\le 2\|B\|\,\mathbb{1}_{Z\cap Y\neq\emptyset}$

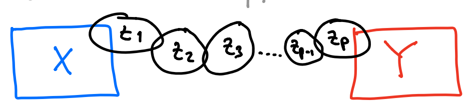
→ $g_B(X,t)\le g_B(X,0) + 2\sum_{Z\cap X\neq\emptyset}\|h_Z\|\int_0^{|t|} g_B(Z,s)\,ds$

$$\le 2\sum_{Z_1\cap X\neq\emptyset}\|h_{Z_1}\|\int_0^{|t|}ds_1\Big[\underbrace{g_B(Z_1,0)}_{\le 2\|B\|\,\mathbb{1}_{Z_1\cap Y\neq\emptyset}} + 2\sum_{Z_2\cap Z_1\neq\emptyset}\|h_{Z_2}\|\int_0^{s_1}ds_2\, g_B(Z_2,s_2)\Big]$$

$$\le 2\|B\|\frac{2|t|}{1!}\sum_{\substack{Z_1\cap X\neq\emptyset\\ Z_1\cap Y\neq\emptyset}}\|h_{Z_1}\| + 2\|B\|\frac{(2|t|)^2}{2!}\sum_{Z_1\cap X\neq\emptyset}\|h_{Z_1}\|\sum_{\substack{Z_2\cap Z_1\neq\emptyset\\ Z_2\cap Y\neq\emptyset}}\|h_{Z_2}\| + \dots$$

use $\int_0^{|t|}ds_1\int_0^{s_1}ds_2\cdots\int_0^{s_{p-1}}ds_p = \frac{|t|^p}{p!}$ = volume of $\{0\le s_p\le s_{p-1}\le\dots\le s_1\le|t|\}$

→ after p steps, we are summing over



Proof step 3 : Use the hypothesis to bound each term

→ 1^{st} term: $\sum_{\substack{Z\cap X\neq\emptyset\\ Z\cap Y\neq\emptyset}}\|h_Z\| \le \sum_{x\in X}\sum_{\substack{Z\ni x\\ Z\cap Y\neq\emptyset}}\|h_Z\| \le \sum_{x\in X}\Big(\sum_{\substack{Z\ni x\\ Z\cap Y\neq\emptyset}}\|h_Z\|\,e^{\overbrace{c\,\mathrm{diam}(Z)}^{\ge d(x,Y)}}\underbrace{|Z|}_{\ge 1}\Big)e^{-c\,d(x,Y)}$

$$\le \|\underline{h}\|_c\sum_{x\in X}e^{-c\,d(x,Y)}$$

→ 2^{nd} term: $\sum_{Z_1\cap X\neq\emptyset}\|h_{Z_1}\|\sum_{\substack{Z_2\cap Z_1\neq\emptyset\\ Z_2\cap Y\neq\emptyset}}\|h_{Z_2}\| \le \sum_{x\in X}\sum_{Z_1\ni x}\|h_{Z_1}\|\sum_{i\in Z_1}\underbrace{\sum_{\substack{Z_2\ni i\\ Z_2\cap Y\neq\emptyset}}\|h_{Z_2}\|}_{\le \|\underline{h}\|_c\, e^{-c\overbrace{d(i,Y)}^{\ge d(x,Y)-d(x,i)}}}$

$$\le \|\underline{h}\|_c\sum_{x\in X}\sum_{Z_1\ni x}\|h_{Z_1}\|\sum_{i\in Z_1}e^{-c\,d(x,Y)}e^{c\underbrace{d(x,i)}_{\le \mathrm{diam}(Z_1)}}$$

$$\le \|\underline{h}\|_c\sum_{x\in X}e^{-c\,d(x,Y)}\sum_{Z_1\ni x}\|h_{Z_1}\|\cdot\underbrace{|Z_1|}_{\ge 1}\cdot e^{c\,\mathrm{diam}(Z_1)} \le \|\underline{h}\|_c^2\sum_{x\in X}e^{-c\,d(x,Y)}$$

→ we get $g_B(X,t)\le 2\|B\|\left(\sum_{p=1}^{\infty}\frac{(2\|\underline{h}\|_c|t|)^p}{p!}\right)\underbrace{\sum_{x\in X}}_{|X|\text{ terms}} e^{-c\underbrace{d(x,Y)}_{\ge d(X,Y)}} \le 2\|B\|\,|X|\,e^{-c\,d(X,Y)}\left(e^{2\|\underline{h}\|_c|t|}-1\right)$ ∎