

# COMPATIBILITY OF QUANTUM MEASUREMENTS

AND

## INCLUSION OF FREE SPECTRAHEDRA

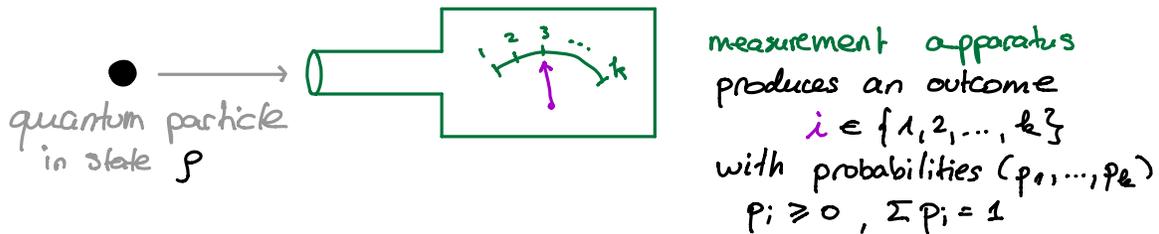
joint work with Andreas BLUTH (QMATH, Copenhagen)

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Goal **Relate**  $\left. \begin{array}{l} \text{Compatibility of POVMs in QM} \\ \text{Noise robustness} \end{array} \right\} \equiv \left\{ \begin{array}{l} \text{Inclusion of free spectrahedra} \\ \text{Inclusion constants for } \dots \end{array} \right. \blacklozenge$

### ① Compatibility of quantum measurements

- quantum states  $\equiv$  density matrices  $\{ \rho \in M_d(\mathbb{C}) : \rho \geq 0 \text{ and } \text{Tr } \rho = 1 \}$   
 $\leftarrow$  PSD order
- measurement results in QM are random



Definition A  $k$ -valued **POVM** is a  $k$ -tuple  $A = (A_1, \dots, A_k) \in M_d(\mathbb{C})^k$  such that  $A_i \geq 0 \forall i$  and  $\sum_{i=1}^k A_i = I_d$ . When measuring a q. state  $\rho$  with  $A$  we obtain outcome  $i$  with probability  $p_i = \text{Tr}[\rho A_i]$ .

Example  $k=2$   $A = (E, I-E)$  for some  $0 \leq E \leq I$ .  $E$  is called a **quantum effect**.

$$\left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) ; \left( \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) ; \left( \frac{1}{3} I, \frac{2}{3} I \right)$$

$1 \times 1$     $1 \times 1$     $1 \times 1$     $1 \times 1$     $\uparrow$   
 outcome does not depend on the state

Compatibility Can two POVMs  $A, B$  be measured "at the same time" ?

Definition Two POVMs  $A, B$  are said to be **compatible** if there exists a POVM  $C$  called a **joint POVM** such that  $\forall$  quantum state  $\rho$ , the probabilities  $p = [\text{Tr}(A_i \rho)]_{i=1}^k$  and  $q = [\text{Tr}(B_j \rho)]_{j=1}^l$  can be obtained as **classical post-processing** of  $r = [\text{Tr}(C_x \rho)]$

Fact  $A, B$  compatible if  $\exists C_{ij} \geq 0, i \in [k], j \in [l]$  s.t.  $\begin{cases} \forall i \sum_j C_{ij} = A_i \\ \forall j \sum_i C_{ij} = B_j \end{cases}$  ← marginals of  $C$

Two effects  $E, F$  are compatible if  $\exists X_{11}, X_{12}, X_{21}, X_{22} \geq 0$  s.t.  $\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = E$   
 $\begin{matrix} \text{''} \\ \text{''} \\ \text{''} \\ \text{''} \end{matrix} \begin{matrix} \\ \\ F \\ I-F \end{matrix}$

This can be generalized to  $g$ -tuples of POVMs:  $A^{(1)}, A^{(2)}, \dots, A^{(g)}$  are compatible if

$\exists C = (C_{i_1 \dots i_g})$  s.t.  $\forall n \in [g], \forall j \in [k_n], A_j^{(n)} = \sum_{i_1, \dots, i_{n-1}, i_{n+1}, \dots, i_g} C_{i_1 \dots i_{n-1} j i_{n+1} \dots i_g}$   
all indices but the  $n$ -th

Examples

①  $A = (a_1 I, \dots, a_k I)$  and  $B = (b_1 I, \dots, b_l I)$  are compatible:  $C_{ij} := a_i b_j I$   
 ↑ these are called trivial POVMs

② If  $[A_i, B_j] = 0 \forall i, j \Rightarrow A$  and  $B$  are compatible

take  $C_{ij} := A_i B_j = A_i^{1/2} B_j A_i^{1/2} \geq 0$ .

③  $(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$  and  $(\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix})$  are not compatible

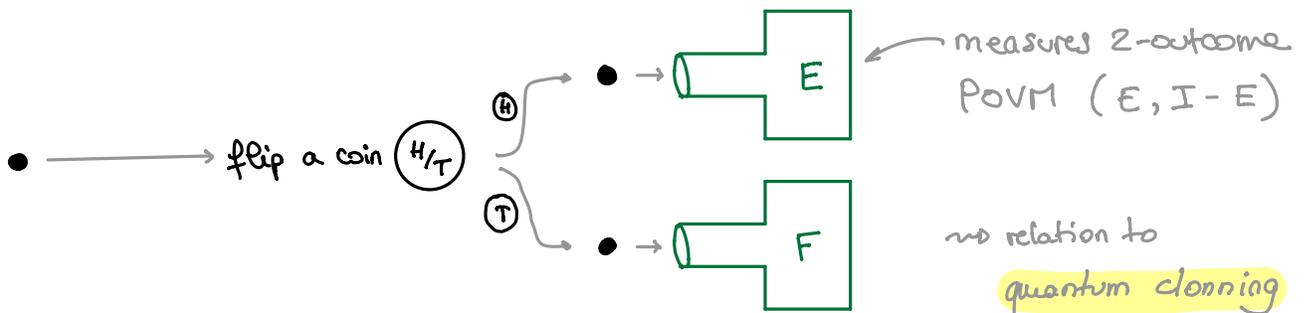
$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$   
 $\Rightarrow X_{11} = 0$ , same for all the others ↯  
 $\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Fact POVMs can be made compatible by adding noise, i.e. mixing trivial POVMs

$(E, I-E) \mapsto s(E, I-E) + (1-s)(\frac{I}{2}, \frac{I}{2})$  ← white noise  
 or  $E \mapsto sE + (1-s)\frac{I}{2}$

Proposition Given any pair of effects  $0 \leq E, F \leq I_d$ , the POVMs  $\frac{1}{2}(E, I-E) + \frac{1}{2}(\frac{I}{2}, \frac{I}{2})$

and  $\frac{1}{2}(F, I-F) + \frac{1}{2}(\frac{I}{2}, \frac{I}{2})$  are compatible



→ the procedure above is equivalent to measuring at the same time the noisy

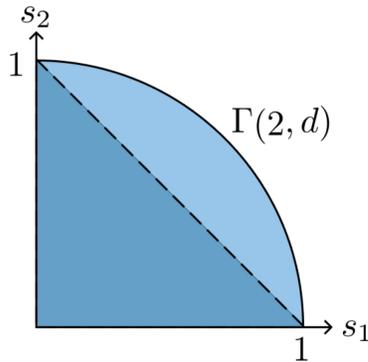
effects  $E' = \frac{1}{2}E + \frac{1}{2}\frac{I}{2}$  and  $F' = \frac{1}{2}F + \frac{1}{2}\frac{I}{2}$

→  $C_{11} = \frac{1}{4}(E+F)$ ;  $C_{12} = \frac{1}{4}(E+I-F)$ ;  $C_{21} = \frac{1}{4}(I-E+F)$ ;  $C_{22} = \frac{1}{4}(I-E+I-F)$

Definition The **compatibility region** for  $g$  effects in  $\Gamma_d$  is the set

$$\Gamma(g, d) := \left\{ s \in [0, 1]^g : \forall \text{ effects } 0 \leq E_1, \dots, E_g \leq I_d, s_i E_i + (1-s_i) \frac{I_d}{2} \text{ are comp.} \right\}$$

- Facts
- $(0, \dots, 0, 1, 0, \dots, 0) \in \Gamma(g, d)$  : measure  $(E_n, I_d - E_n)$ , flip a coin for the others  
↑ position  $n$
  - $\Gamma(g, d)$  is convex. In particular  $(\frac{1}{g}, \dots, \frac{1}{g}) \in \Gamma(g, d)$
  - $g=2$



$\Gamma(2, d)$  is a quarter-circle  $\forall d$

Goal Compute  $\Gamma(g, d)$  for all values of  $g, d$   
↑ # of PVMs      ↑ Hilbert space dimension

## ② Free spectrahedra

Definition A **spectrahedron** is a set

defined by a linear matrix ineq.

$$S = \left\{ x \in \mathbb{R}^g : \sum_{i=1}^g x_i A_i \leq I_d \right\}$$

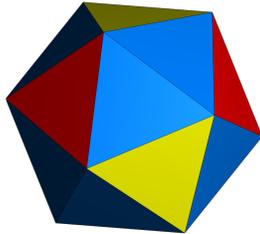
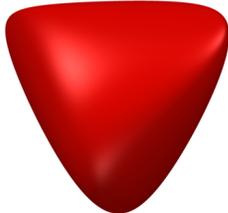
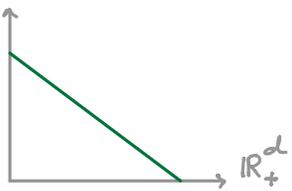
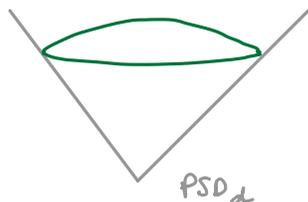
where  $A_1, \dots, A_g \in M_d^{sa}(\mathbb{C})$

→ Spectrahedra are **convex** and **semi-algebraic**  
 $\{x : P_j(x) \geq 0, j=1, \dots, J\}$

Examples → set of density matrices

→ the **elliptope** → correlation mat.

$$\left\{ x \in \mathbb{R}^3 : \begin{bmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{bmatrix} \geq 0 \right\}$$

	
<b>POLYHEDRON</b>	<b>SPECTRAHEDRON</b>
→ feasible set of LP 	→ feasible set of SDP 

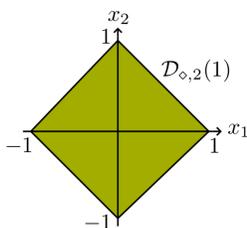
Definition Let  $A \in (M_d^{\text{sa}}(\mathbb{C}))^g$ . The **free spectrahedron at level  $n$**  is the set

$$\mathcal{D}_A(n) := \left\{ X \in (M_n^{\text{sa}}(\mathbb{C}))^g : \sum_{i=1}^g A_i \otimes X_i \leq I_{dn} \right\}$$

The **free spectrahedron** is the union of these levels:  $\mathcal{D}_A := \bigcup_{n \in \mathbb{N}} \mathcal{D}_A(n)$ .

Example The **matrix diamond**  $\mathcal{D}_{\diamond, g}(n) := \left\{ X \in (M_n^{\text{sa}}(\mathbb{C}))^g : \sum_{i=1}^g \varepsilon_i X_i \leq I_n \quad \forall \varepsilon \in \{\pm 1\}^g \right\}$

For  $g=2$ :



This can be obtained

$$\text{from } A_1 = \text{diag}(1, 1, -1, -1)$$

$$A_2 = \text{diag}(1, -1, 1, -1)$$

Def An **operator system** is a linear subspace  $\mathcal{L} \subseteq M_d(\mathbb{C})$  s.t.  $I \in \mathcal{L}$  and  $\mathcal{L} = \mathcal{L}^*$ .

A linear map  $\Phi: \mathcal{L} \rightarrow M_d(\mathbb{C})$  is called  **$n$ -positive** if  $\Phi \otimes \text{id}_m: M_n(\mathcal{L}) \rightarrow M_n(M_d)$

is a positive map.  $\Phi$  is called **completely positive** if it is  $n$ -positive  $\forall n \in \mathbb{N}$ .

Theorem [Helton, Klep, McCullough '13] Let  $A \in (M_d^{\text{sa}}(\mathbb{C}))^g, B \in (M_d^{\text{sa}}(\mathbb{C}))^g$  s.t.  $\mathcal{D}_A(1)$  is bounded.

The unital linear map  $\Phi: \text{span}\{I, A_1, \dots, A_g\} \rightarrow M_d^{\text{sa}}(\mathbb{C})$  given by  $\Phi(A_i) = B_i \quad \forall i \in [g]$

is  **$n$ -positive** iff  $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$ . In particular,  $\Phi$  is **CP** iff  $\mathcal{D}_A \subseteq \mathcal{D}_B$ .

Proof idea Let  $\Phi$  be  $n$ -positive and  $X = (X_1, \dots, X_g) \in \mathcal{D}_A(n)$ . We have:

$$0 \leq [\Phi \otimes \text{id}_m] \left( I - \sum_{i=1}^g A_i \otimes X_i \right) = I - \sum_{i=1}^g B_i \otimes X_i \Rightarrow X \in \mathcal{D}_B(n)$$

$\rightarrow$  Obviously  $\mathcal{D}_A \subseteq \mathcal{D}_B \Rightarrow \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$ . **How about the reverse inclusion?**

Definition The **inclusion set** for the matrix diamond is

$$\Delta(g, d) = \left\{ s \in [0, 1]^g : \forall B \in (M_d^{\text{sa}}(\mathbb{C}))^g, \mathcal{D}_{\diamond, g}(1) \subseteq \mathcal{D}_B(1) \Rightarrow s \cdot \mathcal{D}_{\diamond, g} \subseteq \mathcal{D}_B \right\}$$

$\uparrow$   
 $\{(s_1 X_1, \dots, s_g X_g) : \dots\}$

Facts about  $\Delta(g, d)$

• [Helton, Klep, McCullough, Schweighofer '19]:  $\frac{1}{2d}(1, \dots, 1) \in \Delta(g, d)$

• [Passer, Stahlhut, Solel '18]:  $\mathcal{QC}_g := \{s \in [0, 1]^g : \sum s_i^2 \leq 1\} \subseteq \Delta(g, d)$

$\uparrow$  quarter-circle

### ③ Connecting the two problems

**Theorem** [Bluhm-N '19] let  $E \in (M_d^{\text{sa}}(\mathbb{C}))^g$  be a  $g$ -tuple of self-adjoint matrices. Then:

1.  $\mathcal{D}_{\diamond, g}(1) \subseteq \mathcal{D}_{2E-I}(1)$  iff  $E_1, \dots, E_g$  are quantum effects. (level 1)

2.  $\mathcal{D}_{\square, g} \subseteq \mathcal{D}_{2E-I}$  iff  $E_1, \dots, E_g$  are compatible. (free level)

3.  $\mathcal{D}_{\diamond, g}(n) \subseteq \mathcal{D}_{2E-I}(n)$  for  $n \in [d]$  iff  $\forall$  isometry  $V: \mathbb{C}^n \rightarrow \mathbb{C}^d$ , the compressions  $V^* E_1 V, \dots, V^* E_g V$  are compatible quantum effects. (interm. level)

We also have  $\Gamma(g, d) = \Delta(g, d)$ .

#### Proof ideas

- level 1: consider the extreme points  $\pm e_i$  of the matrix diamond:

$$\pm e_i \in \mathcal{D}_{2E-I}(1) \Leftrightarrow \pm(2E_i - I) \leq I \Leftrightarrow E_i \leq I \text{ and } -E_i \leq 0 \text{ i.e. } E_i \text{ effect}$$

- Free inclusion holds iff the unital map

$$\Phi: I_2^{\otimes(i-1)} \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \otimes I_2^{\otimes(g-i)} \mapsto 2E_i - I_d \text{ is completely positive}$$

- Arveson's extension theorem**:  $\Phi$  has a CP extension  $\tilde{\Phi}$  to  $\mathbb{C}^{2^g}$ .

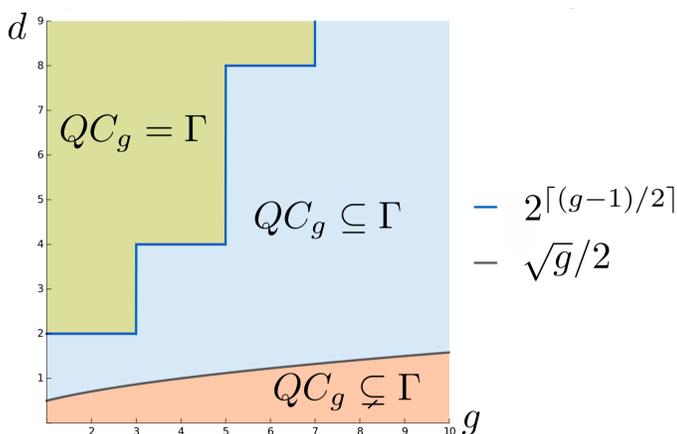
- $\mathbb{C}^{2^g}$  is a commutative algebra:  $\tilde{\Phi}$  CP  $\Leftrightarrow \tilde{\Phi}$  is positive

- Consider the following basis  $x_\eta$  of  $\mathbb{C}^{2^g}$ :

$$x_\eta(\varepsilon) = \prod_{\varepsilon=\eta}^{\text{coord.}} \varepsilon \geq 0$$

- $X_\eta := \tilde{\Phi}(x_\eta)$  is a joint POVM for  $E_1, \dots, E_g$  iff  $\tilde{\Phi}$  positive.

#### What is known about $\Gamma(g, d) = \Delta(g, d)$



- $QC_g = \{s \in [0, 1]^g : \sum_{i=1}^g s_i^2 \leq 1\} \subseteq \Gamma(g, d) \forall d$

- "top region": there exists a  $g$ -tuple of maximally incompatible effects

- "bottom region": cannot have equality because of  $\frac{1}{2d}(1, \dots, 1) \in \Gamma(g, d)$

- "central region": open

## Maximally incompatible effects [PSS'18]: use families of anti-commuting, s.a. unitary op.

For  $d=2^k$ , there exist  $2k+1$  such operators in  $M_d(\mathbb{C})$  [Newman '32]

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$F_1^{(1)} = \sigma_x; \quad F_2^{(1)} = \sigma_y; \quad F_3^{(1)} = \sigma_z;$$

$$F_i^{(k+1)} = \sigma_x \otimes F_i^{(k)} \quad \forall i \in [2k+1]; \quad F_{2k+2}^{(k+1)} = \sigma_y \otimes I_{2^k}; \quad F_{2k+3}^{(k+1)} = \sigma_z \otimes I_{2^k}$$

### ④ Generalizations

→ we have discussed compatibility of quantum effects, i.e. 2-outcome POVMs (E, I-E)

→ elements of the matrix diamond can be seen as incompatibility witnesses

→ in the general case, we have POVMs  $A^{(n)}$  with  $k_n$  outcomes.

→ the Theorem holds, with the matrix diamond replaced by the matrix jewel

$$\cdot \text{ at level 1: } \mathcal{D}_{\mathcal{J}, (k_1, \dots, k_g)}^{(1)} := \mathcal{D}_{\mathcal{J}, k_1}^{(1)} \oplus \dots \oplus \mathcal{D}_{\mathcal{J}, k_g}^{(1)}$$

where  $\mathcal{D}_{\mathcal{J}, k}^{(1)}$  is isomorphic to a  $(k-1)$ -simplex.

and " $\oplus$ " is the direct sum of convex sets:

$$K_1 \oplus K_2 := \text{conv}(\{(x, 0) : x \in K_1\} \cup \{(0, y) : y \in K_2\})$$

not to be confused with  $K_1 \times K_2 = \{(x, y) : x \in K_1, y \in K_2\}$

$$\cdot \text{ extend this to the free level: } \mathcal{D}_{\mathcal{J}, \underline{k}} := \widehat{\mathcal{D}}_{\mathcal{J}, k_1} \widehat{\oplus} \dots \widehat{\oplus} \widehat{\mathcal{D}}_{\mathcal{J}, k_g}$$

→ the matrix jewel has fewer symmetries than the diamond. Many "classical" results do not apply

→ we have very loose bounds, based mainly on symmetrization arguments or on

QIT techniques such as  $\left\{ \begin{array}{l} \text{quantum cloning (for LB)} \\ \text{incompatibility criteria (for UB)} \end{array} \right.$

→ the Theorem also holds in very different settings, such as

work in progress  $\left\{ \begin{array}{l} \bullet \text{ compatibility of quantum channels} \\ \bullet \text{ GPTs (generalized probabilistic theories)} \end{array} \right.$

↳ min- and max- tensor products, tensor norms.