Multipartite entanglement detection via projective tensor norms

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Talk outline

Entanglement in quantum theory

Tensor norms in Banach spaces

Entanglement testers

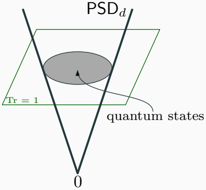
The power of testers

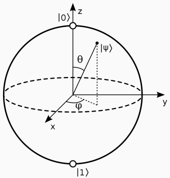
Entanglement in quantum theory

Quantum states — single systems

- Pure quantum states of one particle: unit norm vectors inside a complex Hilbert space $\mathcal{H} = \mathbb{C}^d$
- Mixed quantum states (or density matrices): positive semidefinite matrices of unit trace $\rho \geq 0$, $\operatorname{Tr} \rho = 1$. Importantly, the set of quantum states is not a simplex (as in classical probability)
- In other words

$$\text{quantum states} = \mathrm{PSD}_d \cap \{\operatorname{Tr} = 1\}$$





- Metric point of view: $\{X \in \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C}) \,:\, \operatorname{Tr} X = \|X\|_{S_1^d} = 1\} \rightsquigarrow S_1^d$
- Extreme points: pure states $P_x = |x\rangle\langle x|$, with $x \in \mathbb{C}^d$, $\|x\| = 1$

More systems

- More particles $\rightsquigarrow$ tensor product of the Hilbert spaces $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_m}$
- Notion of positivity?

$$\mathrm{PSD}_{d_1} \otimes_{\min} \mathrm{PSD}_{d_2} \subseteq \mathbb{R}_+ \cdot \{\text{quantum states on } \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}\} \subseteq \mathrm{PSD}_{d_1} \otimes_{\max} \mathrm{PSD}_{d_2}$$

- Nature chose intermediate setting

$$\{\text{quantum states on } \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}\} = \mathrm{PSD}_{d_1 d_2} \cap \{\mathrm{Tr} = 1\}$$

- Elements in the min tensor product are called separable

$$\rho = \sum_{k=1}^{r} p_k \rho_k^1 \otimes \rho_k^2$$

with $\rho_k^{1,2} \in \mathrm{PSD}_{d_{1,2}}$ and $\mathrm{Tr}\, \rho_k^{1,2} = 1$ and p a probability distribution

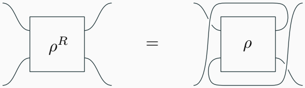
- Separable states can be prepared locally (+ shared randomness)
- Non-separable states $\mathrm{PSD}_{d_1 d_2} \setminus \mathrm{PSD}_{d_1} \otimes_{\min} \mathrm{PSD}_{d_2}$ are called entangled
- Example: the maximally entangled state $\omega_d := \frac{1}{d} \sum_{i,j=1}^{d} |e_i \otimes e_i\rangle\langle e_j \otimes e_j|$
- Pure state $\rho = |x\rangle\langle x|$ is separable iff $x = x^1 \otimes x^2$

Entanglement vs separability

- Deciding whether a given state is separable or entangled (i.e. membership in PSD $\otimes_{\min}$ PSD) is NP-hard [Gurvits]
- Necessary conditions for separability (or sufficient conditions for entanglement), which are computationally efficient
- Partial transposition criterion (PPT) [Peres, Horodecki^3]: given ρ bipartite quantum state ($\top$ is the transposition map)

$$\rho \text{ separable} \implies [\mathrm{id} \otimes \top](\rho) \geq 0$$

- PPT is also sufficient in dimensions 2×2 and 2×3 [Woronowicz]
- Realignment criterion [Chen, Wu; Rudolph]: define the realignment ρ^R of $\rho \in \mathcal{M}_d \otimes \mathcal{M}_d$ as $\rho^R_{ij,kl} = \rho_{ik,jl}$



- ρ separable $\implies \|\rho^R\|_{S_1^{d^2}} \leq 1$
- Both PPT and realignment criteria detect all pure entangled states

Tensor norms in Banach spaces

Injective and projective tensor norms

Definition

Consider m Banach spaces $A_1, \dots, A_m$. For a tensor $x \in A_1 \otimes \cdots \otimes A_m$, we define its projective tensor norm

$$\|x\|_\pi := \inf \left\{ \sum_{k=1}^{r} \|a_k^1\| \cdots \|a_k^m\| \ : \ a_k^i \in A_i,\, x = \sum_{k=1}^{r} a_k^1 \otimes \cdots \otimes a_k^m \right\}$$

and its injective tensor norm

$$\|x\|_\varepsilon := \sup \left\{ \langle \alpha^1 \otimes \cdots \otimes \alpha^m, x \rangle \ : \ \alpha^i \in A_i^*,\ \|\alpha^i\| \leq 1 \right\}$$

- The projective and injective norms are examples of tensor norms (aka reasonable cross-norms):

$$\|a^1 \otimes \cdots \otimes a^m\|_\pi = \|a^1 \otimes \cdots \otimes a^m\|_\varepsilon = \|a^1\| \cdots \|a^m\| \quad + \text{same for dual norms}$$

- Basic examples: $\|\cdot\|_{S_1^d} = \|\cdot\|_{\ell_2^d \otimes_\pi \ell_2^d}$ and $\|\cdot\|_{S_\infty^d} = \|\cdot\|_{\ell_2^d \otimes_\varepsilon \ell_2^d}$
- For any other tensor norm $\|\cdot\|$ on $A_1 \otimes \cdots \otimes A_m$, we have

$$\forall\, x \in A_1 \otimes \cdots \otimes A_m, \quad \|x\|_\varepsilon \leq \|x\| \leq \|x\|_\pi$$

Entanglement with tensor norms

Proposition

A pure quantum state $\psi \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_m}$, $\|\psi\|_2 = 1$, *is separable iff*

$$\|\psi\|_{\ell_2^{d_1} \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_2^{d_m}} = \|\psi\|_{\ell_2^{d_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{d_m}} = 1$$

- The geometric measure of entanglement:

$$G(\psi) := -\log \sup_{\varphi_i \in H_i,\ \|\varphi_i\|=1} |\langle \varphi_1 \otimes \cdots \otimes \varphi_m, \psi \rangle|^2 = -2 \log \|\psi\|_\varepsilon$$

Theorem

For a multipartite mixed quantum state $\rho \in \mathcal{M}_{d_1}(\mathbb{C}) \otimes \cdots \otimes \mathcal{M}_{d_m}(\mathbb{C})$, $\rho \geq 0$, $\operatorname{Tr} \rho = 1$, *the following assertions are equivalent:*

- ρ *is separable*
- $\|\rho\|_{S_{1,sa}^{d_1} \otimes_\pi \cdots \otimes_\pi S_{1,sa}^{d_m}} = 1$
- $\|\rho\|_{S_1^{d_1} \otimes_\pi \cdots \otimes_\pi S_1^{d_m}} = 1$

Entanglement testers

Entanglement testers

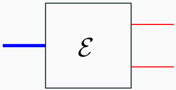
Definition

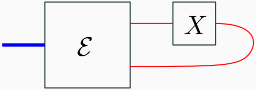
To a n-tuple of matrices $(E_1, \ldots, E_n) \in \mathcal{M}_d(\mathbb{C})^n$, we associate the linear map

$$\mathcal{E} : \mathcal{M}_d(\mathbb{C}) \to \mathbb{C}^n$$
$$X \mapsto \sum_{k=1}^{n} \langle E_k, X\rangle |k\rangle$$

where $\{|k\rangle\}_{k=1}^n$ is some orthonormal basis of $\mathbb{C}^n$. The map $\mathcal{E}$ is called an entanglement tester if

$$\|\mathcal{E}\|_{S_1^d \to \ell_2^n} = 1$$





- The main idea of this work is to use m testers $\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_m$ to embed the projective tensor product of S_1 spaces inside the projective tensor product of the (simpler, commutative) ℓ_2 spaces
- In other words, we reduce the problem of multipartite mixed entanglement to that of multipartite pure entanglement

Detecting entanglement

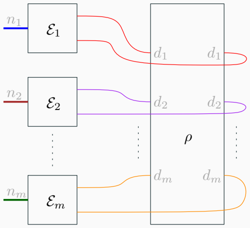
Proposition

If $\mathcal{E}_1, \dots, \mathcal{E}_m$ are entanglement testers, then, for any multipartite quantum state ρ, we have

$$\rho \text{ separable} \iff \|\rho\|_{S_1^{d_1} \otimes_\pi \cdots \otimes_\pi S_1^{d_m}} = 1 \implies \|\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_m(\rho)\|_{\ell_2^{n_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{n_m}} \leq 1$$

Reciprocally, we have the following entanglement criterion:

$$\|\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_m(\rho)\|_{\ell_2^{n_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{n_m}} > 1 \implies \rho \text{ is entangled}$$



- We reduce the evaluation of the tensor norm

$$S_1^{d_1} \otimes_\pi \cdots \otimes_\pi S_1^{d_m} \cong (\ell_2^{d_1} \otimes_\pi \ell_2^{d_1}) \otimes_\pi \cdots \otimes_\pi (\ell_2^{d_m} \otimes_\pi \ell_2^{d_m}) \qquad [2m \text{ factors}]$$

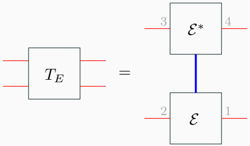
to that of

$$\ell_2^{n_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{n_m} \qquad [m \text{ factors}]$$

Equivalent testers

To a tester $\mathcal{E}$, we associate its test operator $T_E \in \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$

$$T_E = \sum_{k=1}^{n} E_k \otimes E_k^*$$



Definition

Two testers $\mathcal{E}, \mathcal{F} : S_1^d \to \ell_2^n$ are called equivalent if there exists a unitary operator $U \in \mathcal{U}(n)$ such that, for all $X \in \mathcal{M}_d(\mathbb{C})$, we have

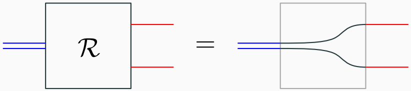
$$\mathcal{F}(X) = U\mathcal{E}(X).$$

Proposition

Two testers $\mathcal{E}, \mathcal{F} : S_1^d \to \ell_2^n$ are equivalent if and only if they have the same test operator $T_E = T_F$.

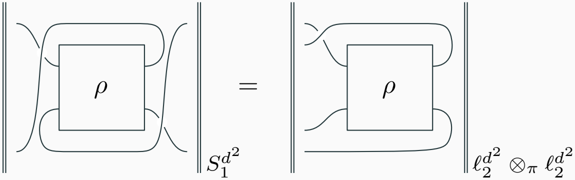
Example 1: Realignment

- Recall the realignment criterion: $\|\rho^R\|_{S_1^{d^2}} > 1 \implies \rho$ entangled
- Let $\mathcal{R} = \mathrm{id} : \mathcal{M}_d(\mathbb{C}) \to \mathbb{C}^{d^2}$. $\mathcal{R}$ is a tester: $\|\mathcal{R}\|_{S_1^d \to \ell_2^{d^2} \cong S_2^d} = 1$



- We have

$$\rho^R = [\mathcal{R} \otimes \mathcal{R}](\rho)$$



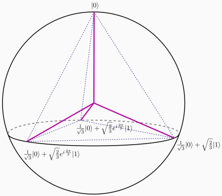
- Hence, the realignment criterion corresponds to the $\mathcal{R} \otimes \mathcal{R}$ tester
- Natural generalization to the multipartite setting: $\mathcal{R}^{\otimes m}$

Example 2: SIC POVM

- A spherical 2-design $\{|x_k\rangle\}_{k=1}^N$ is a finite subset of the unit sphere of $\mathbb{C}^d$ having the same first 2 moments as the Haar measure

$$\frac{1}{N}\sum_{k=1}^{N}|x_k\rangle\langle x_k|^{\otimes 2} = \frac{P_{sym}}{d(d+1)/2}$$

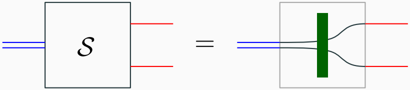
- Spherical 2-designs with $N = d^2$ are known as SIC POVMs. we have $|\langle x_i, x_j\rangle|^2 = 1/(d+1)$ for $i \neq j$



- Existence is conjectured in every dimension, proven for $d = 1, \ldots, 16, 19, 24, 35, 48$

- Any 2-design with d^2 elements defines a tester

$$\mathcal{S} : X \mapsto \sqrt{\frac{d+1}{2d}}\sum_{k=1}^{d^2}\langle x_k|X|x_k\rangle|k\rangle$$



The power of testers

Perfect testers

Definition

A tester $\mathcal{E} : S_1^d \to \ell_2^n$ is called perfect if, for any pure states $\varphi, \chi \in \mathbb{C}^d \otimes \mathbb{C}^d$, at least one of them entangled,

$$\|\mathcal{E}^{\otimes 2}(|\varphi\rangle\langle\chi|)\|_{\ell_2^n \otimes_\pi \ell_2^n} > 1$$

Theorem

For a linear map $\mathcal{E} : S_1^d \to \ell_2^n$ the following statements are equivalent:

- *$\mathcal{E}$ is a perfect tester*
- *The norm $\|\mathcal{E}\|_{S_1^d \to \ell_2^d} = 1$ is attained at all the extremal points of the unit ball of S_1^d: for all unit vectors $x, y \in \mathbb{C}^d$ we have $\|\mathcal{E}(|x\rangle\langle y|)\|_2 = 1$*
- *$\mathcal{E}$ is an isometry $S_2^d \to \ell_2^n$*

- The realignment tester $\mathcal{R}$ is perfect, but the SIC POVM map $\mathcal{S}$ is not...

Actually...

Definition

A $\mathbb{R}$-linear map $\mathcal{F} : \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C}) \to \mathbb{C}^n$ is called an $\mathbb{R}$-tester if

$$\|\mathcal{F}\|_{S_{1,\mathrm{sa}}^d \to \ell_2^n} = 1$$

An $\mathbb{R}$-tester $\mathcal{F} : S_1^{d,\mathrm{sa}} \to \ell_2^n$ is called $\mathbb{R}$-perfect if, for any pure entangled state $\varphi \in \mathbb{C}^d \otimes \mathbb{C}^d$,

$$\|\mathcal{F}^{\otimes 2}(|\varphi\rangle\langle\varphi|)\|_{\ell_2^n \otimes_\pi \ell_2^n} > 1$$

Theorem

For an $\mathbb{R}$-linear map $\mathcal{F} : S_1^{d,sa} \to \ell_2^n$, the following are equivalent:

- *$\mathcal{F}$ is an $\mathbb{R}$-perfect tester*
- *The norm $\|\mathcal{F}\|_{S_1^{d,sa} \to \ell_2^n} = 1$ is attained at all the extremal points of the unit ball of $S_1^{d,sa}$: for all unit vectors $x \in \mathbb{C}^d$ we have $\|\mathcal{E}(|x\rangle\langle x|)\|_2 = 1$*

- The SIC POVM map $\mathcal{S}$ is a perfect $\mathbb{R}$-tester

Symmetric testers

- A tester $\mathcal{E} : S_1^d \to \ell_2^n$ is called symmetric if its test operator is a linear combination of the sym. and the anti-sym. projections $T_E = \alpha F + \beta I$
- The matrix T_E above is a valid test operator iff $\alpha \geq 0$ and $\beta \geq -\alpha/d$
- Recall $\mathcal{R} \rightsquigarrow (\alpha, \beta) = (1, 0)$ and $\mathcal{S} \rightsquigarrow (\alpha, \beta) = (1/2, 1/2)$

Proposition

Consider an arbitrary bipartite unit vector $\varphi \in \mathbb{C}^d \otimes \mathbb{C}^d$, with Schmidt decomposition $|\varphi\rangle = \sum_{i=1}^r \sqrt{\lambda_i} |e_i f_i\rangle$. Let $\mathcal{E} : S_1^d \to \ell_2^n$ be a symmetric tester as above. We have

$$\left\| \mathcal{E}^{\otimes 2}(|\varphi\rangle\langle\varphi|) \right\|_1 = \alpha + \beta + 2\alpha \sum_{i<j} \sqrt{\lambda_i \lambda_j}$$

- For any pure state $|\varphi\rangle$ we have

$$\left\| \mathcal{R}^{\otimes 2}(|\varphi\rangle\langle\varphi|) \right\|_1 - 1 = 2 \left(\left\| \mathcal{S}^{\otimes 2}(|\varphi\rangle\langle\varphi|) \right\|_1 - 1 \right)$$

- This solves a conjecture from [SAZG18][1]

[1] Shang et al. - *Enhanced entanglement criterion via SIC measurements* - PRA 2018

Realignment vs. SIC POVM

- In [SAZG18] the authors conjectured that any entangled state detected by the realignment criterion in also detected by the SIC POVM criterion

Theorem

For any quantum state ρ on $\mathbb{C}^d \otimes \mathbb{C}^d$, we have

$$\|\mathcal{S}^{\otimes 2}(\rho)\|_{\ell_2^{d^2} \otimes_\pi \ell_2^{d^2}} \geq \frac{\|\mathcal{R}^{\otimes 2}(\rho)\|_{\ell_2^{d^2} \otimes_\pi \ell_2^{d^2}} + 1}{2}$$

- Proof idea: perturbation theory for S_1 norm by non-unitary conjugations

$$S = \sum_{i=1}^{n} \gamma_i |a_i\rangle\langle b_i| \implies \|SXS^*\|_1 \geq \|X\|_1 + \sum_{i=1}^{n} (|\gamma_i|^2 - 1)\langle b_i|X|b_i\rangle$$

- For many families of quantum states, such as
  - isotropic states: $\rho = p\omega_d + (1-p)I/d^2$
  - Werner states $\rho = q\hat{P}_{sym} + (1-q)\hat{P}_{asym}$

  the inequality is saturated

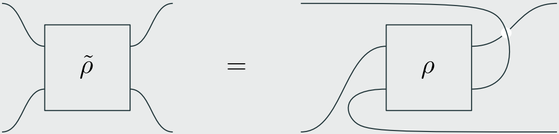
Completeness of the testers for mixed bipartite states

Theorem

Let ρ be an entangled state on $\mathbb{C}^d \otimes \mathbb{C}^d$. Then, there exists a tester $\mathcal{E} : S_1^d \to \ell_2^{d^2}$ such that

$$\left\| \left[\mathcal{E}^\sharp \otimes \mathcal{E}\right](\tilde{\rho}) \right\|_{\ell_2^{d^2} \otimes_\pi \ell_2^{d^2}} > 1,$$

where $\mathcal{E}^\sharp : S_1^d \to \ell_2^{d^2}$ is the tester whose operators are the adjoints of those of $\mathcal{E}$, and $\tilde{\rho}$ is obtained by permuting the legs of ρ as follows:



- Note that $\|\rho\|_{S_1^d \otimes_\pi S_1^d} = \|\tilde{\rho}\|_{S_1^d \otimes_\pi S_1^d}$
- Start from an entanglement witness W such that $\langle W, \rho \rangle > 1$ and $\|W\|_{S_\infty^d \otimes_\varepsilon S_\infty^d} = 1$
- Massage W and take $\mathcal{E} = \sqrt{W'}$

Completeness for multiparite pure states

- Recall

$$\|\varphi\|_{(\ell_2^d)^{\otimes_\varepsilon m}} = \sup_{\|\psi_i\|\leq 1} \langle \psi_1 \otimes \cdots \otimes \psi_m, \varphi \rangle$$

Theorem

For any unit vector $\varphi \in (\mathbb{C}^d)^{\otimes m}$,

$$\left\|\mathcal{R}^{\otimes m}(|\varphi\rangle\langle\varphi|)\right\|_{(\ell_2^{d^2})^{\otimes_\pi m}} \geq \frac{1}{\|\varphi\|_{(\ell_2^d)^{\otimes_\varepsilon m}}}$$

If in addition φ is non-negative (meaning that its coefficients in the canonical basis of $(\mathbb{C}^d)^{\otimes m}$ are all non-negative), then

$$\left\|\mathcal{R}^{\otimes m}(|\varphi\rangle\langle\varphi|)\right\|_{(\ell_2^{d^2})^{\otimes_\pi m}} \geq \frac{1}{\|\varphi\|^2_{(\ell_2^d)^{\otimes_\varepsilon m}}}$$

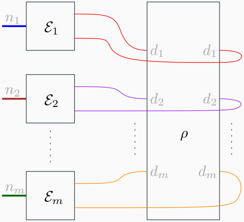
In particular, we have

$$\varphi \text{ entangled } \implies \left\|\mathcal{R}^{\otimes m}(|\varphi\rangle\langle\varphi|)\right\|_{(\ell_2^{d^2})^{\otimes_\pi m}} > 1$$

The take-home slide

- Mixed quantum state: $\rho \in \mathcal{M}_d(\mathbb{C})$, $\rho \geq 0$, $\operatorname{Tr} \rho = 1$
- Multipartite quantum state ρ is separable if

$$\rho = \sum_{k=1}^{r} p_k \rho_k^1 \otimes \cdots \otimes \rho_k^m$$



- Equiv. charact. in terms of projective tensor norm

$$\|\rho\|_{S_1^{d_1} \otimes_\pi \cdots \otimes_\pi S_1^{d_m}} = 1$$

- Entanglement tester: contraction $\mathcal{E} : S_1^d \to \ell_2^n$
- Ent. criterion: $\|[\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_m](\rho)\|_{\ell_2^{n_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{n_m}} > 1 \implies \rho$ entangled
- General framework, encompasses (and extends) many known criteria (realignment $\mathcal{R}$, SIC POVM $\mathcal{S}$). Proof of conjecture $\mathcal{R} \subset \mathcal{S}$
- Testers complete for mixed bipartite states* and pure multipartite states

Open problems

- Completeness for mixed multipartite states
- Imperfect testers $\mathcal{E} : S_1^d \to \ell_2^n$ with $n \ll d^2$