The multiple facets of measurement compatibility in GPTs

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Talk outline

Compatibility in GPTs

Positivity & tensor products

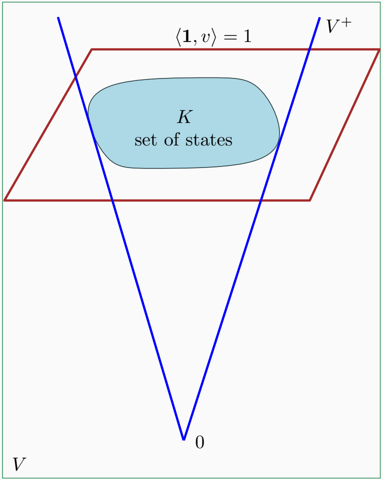
Generalized spectrahedra

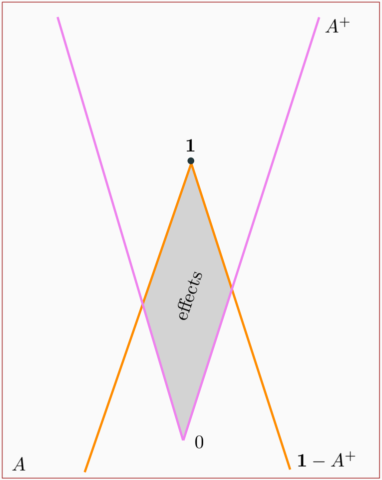
Tensor norms & applications

Martha Mayer Erlebacher - *Three-Faced Figure*

Compatibility in GPTs

General Probabilistic Theories

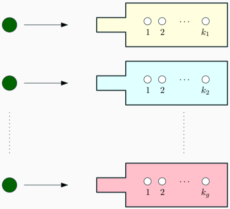




- A GPT is a triple $(V, V^+, \mathbb{1})$, where V is a vector space, $V^+ \subseteq V$ is a cone, and $\mathbb{1}$ is a linear form on V; $A = V^*$, $A^+ = (V^+)^*$, and $\mathbb{1} \in A^+$
- The set of states $K := V^+ \cap \mathbb{1}^{-1}(\{1\})$

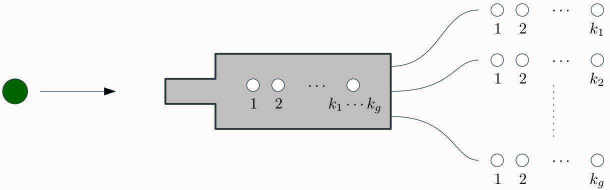
Measurements and compatibility

- A GPT measurement with k outcomes is an affine map $K \to \Delta_k$
- A g-tuple of GPT measurements with $\mathbf{k} = (k_1, \ldots, k_g)$ outcomes are encoded in an affine map $K \to \Delta_{k_1} \times \cdots \times \Delta_{k_g} =: P_{\mathbf{k}}$ (the polysimplex)

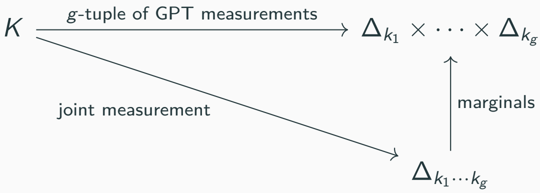


- Measurements $f = (f^{(1)}, \ldots, f^{(g)})$ are compatible if there exists a joint measurement g having $k_1 \cdots k_g$ outcomes such that

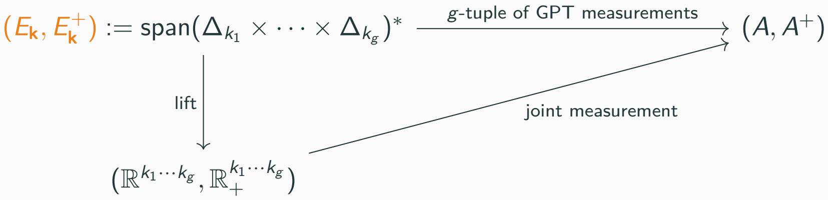
$$\forall x \in [g], \quad \forall i \in [k_x], \qquad f_i^{(x)} = \sum_{j \in [k_1] \times \cdots \times [k_g] \,:\, j(x) = i} g_j$$



Compatibility with diagrams



- Take duals and linearize!



Positivity & tensor products

Bipartite systems in GPTs

- Consider two GPTs $(V_A, V_A^+, \mathbb{1}_A)$ and $(V_B, V_B^+, \mathbb{1}_B)$. Joint system AB?

$$(V_A \otimes V_B, ???, \mathbb{1}_A \otimes \mathbb{1}_B)$$

- Minimal tensor product cone

$$V_A^+ \otimes_{\text{min}} V_B^+ := \left\{ \sum_{i=1}^{r} a_i \otimes b_i \ : \ a_i \in V_A^+, \ b_i \in V_B^+ \right\}$$

- Maximal tensor product cone

$$V_A^+ \otimes_{\text{max}} V_B^+ := \left((V_A^+)^* \otimes_{\text{min}} (V_B^+)^* \right)^*$$

- A tensor cone is anything in between

$$V_A^+ \otimes_{\text{min}} V_B^+ \subseteq C \subseteq V_A^+ \otimes_{\text{max}} V_B^+$$

- In general, a linear map $\Phi : A \to B$ is positive if its associated tensor $\varphi \in A^* \otimes B$ is an element of $(A^+)^* \otimes_{\text{max}} B^+$

- Question: What is $\mathrm{PSD}_{d_A} \otimes_{\text{min}} \mathrm{PSD}_{d_B}$? How about $\mathrm{PSD}_{d_A} \otimes_{\text{max}} \mathrm{PSD}_{d_B}$

Measurements as positive maps

We consider g measurements $f^{(1)}, \dots, f^{(g)}$ having $k_1, \dots, k_g$ outcomes:

$$\forall x \in [g], \quad \forall i \in [k_x],\ f_i^{(x)} \in A^+ \ \text{and} \ \sum_{i=1}^{k_x} f_i^{(x)} = \mathbb{1}$$

Proposition

The following are equivalent

1. *The tuple $f = (f^{(1)}, \dots, f^{(g)})$ consists of GPT measurements*
2. *The following map is positive*

$$\Phi^{(f)} : (E_{\mathbf{k}}, E_{\mathbf{k}}^+) \to (A, A^+)$$
$$\mathbb{1}_{\mathbf{k}} \mapsto \mathbb{1}$$
$$\eta_i^{(x)} \mapsto f_i^{(x)}$$

3. *The associated tensor $\varphi^{(f)} \in (E_{\mathbf{k}}^+)^* \otimes_{\max} A^+$*

$$\eta_i^{(x)} = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{x-1 \text{ times}} \otimes e_i \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{g-x \text{ times}} \qquad x \in [g],\ i \in [k_x - 1]$$

Compatible measurements as entanglement breaking maps

Theorem

The following are equivalent

1. *The tuple* $f = (f^{(1)}, \ldots, f^{(g)})$ *consists of compatible GPT meas.*
2. *The map* $\Phi^{(f)}$ *admits a positive extension* $\tilde{\Phi}^{(f)} : (\mathbb{R}^{\mathbf{k}}, \mathbb{R}^{\mathbf{k}}_{+}) \to (A, A^{+})$
3. *The map* $\Phi^{(f)}$ *is entanglement breaking*
4. *The associated tensor* $\varphi^{(f)} \in (E^{+}_{\mathbf{k}})^{*} \otimes_{\min} A^{+}$

Definition

A positive map $\Phi : (C, C^{+}) \to (D, D^{+})$ is called entanglement breaking if any of the following equivalent conditions holds

- For all (L, L^{+}), $\Phi \otimes \mathrm{id}_{L} : C^{+} \otimes_{\max} L^{+} \to D^{+} \otimes_{\min} L^{+}$ is positive
- The condition above holds for $(L, L^{+}) = (D^{*}, (D^{+})^{*})$
- The associated tensor $\varphi \in (C^{+})^{*} \otimes_{\min} D^{+}$

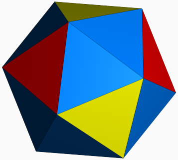
Remark

If $g = 1$, $(E^{+}_{(k)})^{*}$ *is simplicial* $\implies$ $(E^{+}_{(k)})^{*} \otimes_{\min} A^{+} = (E^{+}_{(k)})^{*} \otimes_{\max} A^{+}$

Generalized spectrahedra

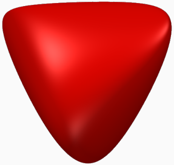
Free spectrahedra

- A polyhedron is defined as the intersection of half-spaces



$$\{x \in \mathbb{R}^g \ : \ \langle h_i, x \rangle \leq 1, \quad \forall i \in [k]\}$$

- A spectrahedron is given by PSD constraints: for $A = (A_1, \ldots, A_g) \in (M_d^{sa})^g$



$$\mathcal{D}_A(1) := \{x \in \mathbb{R}^g \ : \ \sum_{i=1}^{g} x_i A_i \leq I_d\}$$

- Question: What is $\mathcal{D}_{(\sigma_X, \sigma_Y, \sigma_Z)}$?

- A free spectrahedron is the matricization of a spectrahedron

$$\mathcal{D}_A := \bigsqcup_{n=1}^{\infty} \mathcal{D}_A(n) \quad \text{with} \quad \mathcal{D}_A(n) := \{X \in (\mathcal{M}_n^{sa})^g \ : \ \sum_{i=1}^{g} X_i \otimes A_i \leq I_{nd}\}$$

Compatibility in QM via free spectrahedra

- The matrix diamond is the free spectrahedron defined by

$$\mathcal{D}_{\diamondsuit,g} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{sa})^g \ : \ \sum_{i=1}^{g} \varepsilon_i X_i \leq I_n, \quad \forall \varepsilon \in \{\pm 1\}^g\}$$

- To a g-tuple of self-adjoint matrices $f \in (\mathcal{M}_d^{sa})^g$, we associate the free spectrahedron defined by the matrices $2f_i - I_d$:

$$\mathcal{D}_f := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{sa})^g \ : \ \sum_{i=1}^{g} X_i \otimes (2f_i - I_d) \leq I_{nd}\}$$

Theorem

- *The matrices f are quantum effects* $\iff \mathcal{D}_{\diamondsuit,g}(1) \subseteq \mathcal{D}_f(1)$
- *The matrices f are compatible quantum effects* $\iff \mathcal{D}_{\diamondsuit,g} \subseteq \mathcal{D}_f$

- The general (non-dichotomic) version is similar $\rightsquigarrow$ matrix jewel $\mathcal{D}_{\text{jewel},\mathbf{k}}$

Compatibility in GPTs via generalized spectrahedra

- Consider two ordered vector spaces (M, M^+), (L, L^+), and a tensor cone C on $M \otimes L$. A tuple $a \in M^g$ defines a generalized spectrahedron

$$\mathcal{D}_a(L, C) := \{v \in L^g \ : \ \sum_{i=1}^{g} a_i \otimes v_i \in C\}$$

- The GPT jewel is induced by $E_{\mathbf{k}}^+$ (w are some elements related to the η)

$$\mathcal{D}_{\mathrm{GPT}\diamond}(\mathbf{k}; L, L^+) := \mathcal{D}_w(L, E_{\mathbf{k}}^+ \otimes_{\max} L^+)$$

- Shifted versions of GPT elements induce a generalized spectrahedron

$$\mathcal{D}_f(L, L^+) := \mathcal{D}_{\tilde{f}}(L, A^+ \otimes_{\min} L^+)$$

Theorem

- *The elements f are GPT meas.* $\iff \mathcal{D}_{GPT\diamond}(\mathbf{k}; \mathbb{R}, \mathbb{R}_+) \subseteq \mathcal{D}_f(\mathbb{R}, \mathbb{R}_+)$
- *The elements f are compatible* $\iff \mathcal{D}_{GPT\diamond}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_f(V, V^+)$

Compatibility regions and inclusion constants

- Noisy version of GPT measurements (white noise)

$$(s.f)_i^{(x)} = s_x f_i^{(x)} + (1 - s_x)\frac{\mathbb{1}}{k_x}$$

- The set of noise parameters s rendering all measurements compatible is called the compatibility region

$$\Gamma(\mathbf{k}; V, V^+) := \{s \in [0,1]^g \ : \ f \text{ measurements } \implies s.f \text{ compatible}\}$$

- Symmetric version: the compatbility degree

$$\gamma(\mathbf{k}; V, V^+) := \max\{s \ : \ (s, s, \ldots, s) \in \Gamma(\mathbf{k}; V, V^+)\}$$

- The inclusion constants for the GPT jewel

$$\Delta(\mathbf{k}; V, V^+) := \Big\{ s \in [0,1]^g \ : \ \forall a_i^{(x)} \in A, \ \mathcal{D}_{\text{GPT}\diamondsuit}(\mathbf{k}; \mathbb{R}, \mathbb{R}^+) \subseteq \mathcal{D}_a(\mathbb{R}, \mathbb{R}^+)$$
$$\implies (1, s_1^{\times(k_1-1)}, \ldots, s_g^{\times(k_g-1)}) \cdot \mathcal{D}_{\text{GPT}\diamondsuit}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_a(V, V^+) \Big\}$$

Theorem

For all GPTs and all $\mathbf{k}$*, we have* $\Gamma(\mathbf{k}; V, V^+) = \Delta(\mathbf{k}; V, V^+)$

Tensor norms & applications

Injective and projective tensor norms

Definition

Consider m Banach spaces $A_1, \ldots, A_m$. For a tensor $x \in A_1 \otimes \cdots \otimes A_m$, we define its projective tensor norm

$$\|x\|_\pi := \inf \left\{ \sum_{k=1}^{r} \|a_k^1\| \cdots \|a_k^m\| \ : \ a_k^i \in A_i,\ x = \sum_{k=1}^{r} a_k^1 \otimes \cdots \otimes a_k^m \right\}$$

and its injective tensor norm

$$\|x\|_\varepsilon := \sup \left\{ |\langle \alpha^1 \otimes \cdots \otimes \alpha^m, x\rangle| \ : \ \alpha^i \in A_i^*,\ \|\alpha^i\| \leq 1 \right\}$$

- The projective and injective norms are examples of tensor norms (aka reasonable cross-norms):

$$\|a^1 \otimes \cdots \otimes a^m\|_\pi = \|a^1 \otimes \cdots \otimes a^m\|_\varepsilon = \|a^1\| \cdots \|a^m\|$$

$$\|\alpha^1 \otimes \cdots \otimes \alpha^m\|_{\pi*} = \|\alpha^1 \otimes \cdots \otimes \alpha^m\|_{\varepsilon*} = \|\alpha^1\|_* \cdots \|\alpha^m\|_*$$

- For any other tensor norm $\|\cdot\|$ on $A_1 \otimes \cdots \otimes A_m$, we have

$$\forall\ x \in A_1 \otimes \cdots \otimes A_m, \qquad \|x\|_\varepsilon \leq \|x\| \leq \|x\|_\pi$$

- The injective and projective norms are dual to each other

Operator and nuclear norms

- For an operator $X \in \mathcal{M}_d(\mathbb{C})$, the operator norm (or the Schatten ∞ norm) is defined as

$$\|X\|_{S_\infty^d} = \sup_{\|a\|,\|b\|\leq 1} |\langle a, Xb\rangle|$$

- Seeing X as a 2-tensor $\tilde{X} \in \ell_2^d \otimes \ell_2^d$, we have

$$\|X\|_{S_\infty^d} = \sup_{\|a\|,\|b\|\leq 1} |\langle a \otimes b, \tilde{X}\rangle| = \|\tilde{X}\|_{\ell_2^d \otimes_\varepsilon \ell_2^d}$$

- The nuclear norm of X (or the Schatten 1 norm) is dual to the operator norm, so we have

$$\|X\|_{S_1^d} = \|\tilde{X}\|_{\ell_2^d \otimes_\pi \ell_2^d}$$

- This can be seen directly from the SVD: $\|X\|_{S_1^d} = \sum_{i=1}^d \sigma_i$ for

$$X = \sum_{i=1}^d \sigma_i |a_i\rangle\langle b_i| \qquad \Longleftrightarrow \qquad \tilde{X} = \sum_{i=1}^d \sigma_i a_i \otimes b_i$$

for non-negative σ_i and orthonormal bases $\{a_i\}$, $\{b_i\}$

Compatibility and tensor norms

- We shall only consider here the case of dichotomic measurements and centrally symmetric GPTs: K is the unit ball of a norm $\|\cdot\|_{\bar{V}}$

$$V = \mathbb{R}v_0 \oplus \bar{V} \quad \text{and} \quad A = \mathbb{R}\mathbb{1} \oplus \bar{A}$$

Theorem

For dichotomic measurements in centrally symmetric GPTs, we have

$$\gamma(2^{\times g}; V, V^+) = 1/\rho(\ell_\infty^g, \bar{A})$$

where the quantity ρ was introduced in [Aubrun et al '20]

$$\rho(X, Y) = \max_{z \in X \otimes Y} \frac{\|z\|_{X \otimes_\pi Y}}{\|z\|_{X \otimes_\varepsilon Y}}$$

Proposition

In the same setting as before

$$\lim_{g \to \infty} \gamma(2^{\times g}; V, V^+) = 1/\pi_1(\bar{V})$$

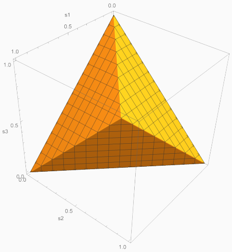
where $\pi_1(\bar{V})$ is the 1-summing norm of the Banach space $\bar{V}$

Applications

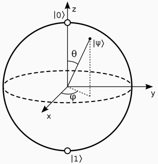
- For the hypercubic GPT $\bar{V} = \ell_\infty^n$, we have

$$\Gamma(2^{\times g}; \ell_\infty^n) = \{s \in [0,1]^g \ : \ \forall I \subseteq [g] \text{ s.t. } |I| \leq n, \sum_{i \in I} s_i \leq 1\}$$

- We have $\gamma(2^{\times g}; \ell_\infty^n) = 1/\min(g, n)$



- Quantum mechanics for $d = 2$ (qubits) is centrally symmetric: $\bar{V} = \ell_2^3$
- It was known that, for $g = 2, 3$, $\gamma(2^{\times g}; \mathrm{QM}_2) = 1/\sqrt{g}$



Proposition

For all $g \geq 4$,

$$0.5 \leq \gamma(2^{\times g}; \mathrm{QM}_2) \leq 1/\sqrt{3} \approx 0.577$$

The take-home slide