Entanglement of generic quantum states

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Talk outline

1. Entanglement in QIT
2. Random quantum states
3. Thresholds for entanglement criteria
4. Random matrices and free probability

Entanglement in QIT

Quantum states and entanglement

- Quantum systems with d degrees of freedom are described by density matrices or mixed states

$$\rho \in \mathcal{M}^{1,+}(\mathbb{C}^d); \qquad \mathrm{Tr}\rho = 1 \text{ and } \rho \geq 0.$$

- Pure states are the particular case of rank one projectors, and correspond to unit vectors $\psi \in \mathbb{C}^d$

$$|\psi\rangle\langle\psi| \in \mathcal{M}^{1,+}(\mathbb{C}^d).$$

  They are the extreme points of the convex body $\mathcal{M}^{1,+}(\mathbb{C}^d)$.

- Two quantum systems: $\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$.
- A mixed state ρ_{AB} is called separable if it can be written as a convex combination of product states

$$\rho_{AB} \in \mathcal{SEP} \iff \rho_{AB} = \sum_i t_i \sigma_i^{(A)} \otimes \sigma_i^{(B)},$$

  with $t_i \geq 0$, $\sum_i t_i = 1$, $\sigma_i^{(A,B)} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_{A,B}})$.

- Non-separable states are called entangled.

Pure state entanglement is easy

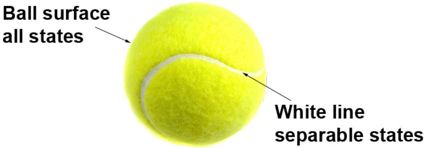
- For pure quantum states, entanglement can be detected and measured. The standard measure of the entanglement of a pure state $x = |x\rangle_{AB}$ is the entropy of entanglement

$$E(x) = -\sum_i s_i(x) \log s_i(x),$$

where $s_i(x)$ are the Schmidt coefficients of x:

$$|x\rangle_{AB} = \sum_i \sqrt{s_i(x)}\, |e_i\rangle_A \otimes |f_i\rangle_B.$$

- $E(x) = 0 \iff x = y \otimes z$.
- All bi-partite quantum pure states have dimension $d_A d_B - 1$, whereas product states have dimension $d_A + d_B - 2$, which is strictly smaller $\implies$ a generic pure state is entangled!



Mixed state entanglement is hard, but...

- Deciding if a given ρ_{AB} is separable is NP-hard. Detecting entanglement for general states is a difficult, central problem in QIT.
- A map $f : \mathcal{M}(\mathbb{C}^d) \to \mathcal{M}(\mathbb{C}^{d'})$ is called
  - positive if $A \geq 0 \implies f(A) \geq 0$;
  - completely positive if $\mathrm{id}_k \otimes f$ is positive for all $k \geq 1$.
- If $f : \mathcal{M}(\mathbb{C}^{d_B}) \to \mathcal{M}(\mathbb{C}^{d_B})$ is CP, then for every state ρ_{AB} one has $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geq 0$.
- If $f : \mathcal{M}(\mathbb{C}^{d_B}) \to \mathcal{M}(\mathbb{C}^{d_B})$ is only positive, then for every separable state ρ_{AB}, one has $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geq 0$.

Entanglement detection via positive, but not CP maps

- Positive, but not CP maps f yield entanglement criteria: given ρ_{AB}, if $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \not\geq 0$, then ρ_{AB} is entangled.
- The following converse holds: if, for all positive, but not CP maps f, $[\mathrm{id}_{d_A} \otimes f](\rho_{AB}) \geq 0$, then ρ_{AB} is separable.

- The transposition map $\Theta(X) = X^\top$ is positive, but not CP. Put

$$\mathcal{PPT} := \{\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \,|\, [\mathrm{id}_{d_A} \otimes \Theta_{d_B}](\rho_{AB}) \geq 0\}.$$

- The reduction map $R(X) = \mathrm{Tr}(X) \cdot I - X$ is positive, but not CP.

$$\mathcal{RED} := \{\rho_{AB} \in \mathcal{M}^{1,+}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \,|\, [\mathrm{id}_{d_A} \otimes R_{d_B}](\rho_{AB}) \geq 0\}.$$

- Both criteria above detect pure entanglement: for $f = \Theta, R$,

$$[\mathrm{id}_{d_A} \otimes f](|\psi\rangle_{AB}\langle\psi|) \geq 0 \iff |\psi\rangle_{AB} \text{ is entangled.}$$

The PPT criterion at work

- Recall the Bell state $\rho_{12} = |\psi\rangle\langle\psi|$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B).$$

- Written as a matrix in $\mathcal{M}_{2\cdot 2}^{1,+}(\mathbb{C})$

$$\rho_{AB} = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) = \frac{1}{2}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

- Partial transposition: transpose each block B_{ij}:

$$[\mathrm{id}_2 \otimes \Theta](\rho_{AB}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- This matrix is no longer positive $\implies$ the state is entangled.

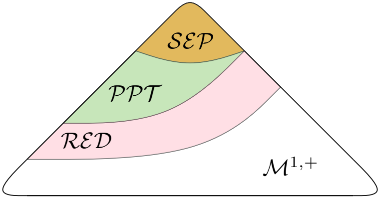
The problem we consider

$\mathcal{M}^{1,+} = \{\rho \,:\, \mathrm{Tr}\rho = 1 \text{ and } \rho \geq 0\}$

$\mathcal{SEP} = \left\{ \sum_i t_i \rho_i^{(A)} \otimes \rho_i^{(B)} \right\}$

$\mathcal{PPT} = \{\rho_{AB} \,:\, [\mathrm{id}_{d_A} \otimes \Theta_{d_B}](\rho_{AB}) \geq 0\}$

$\mathcal{RED} = \{\rho_{AB} \,:\, [\mathrm{id}_{d_A} \otimes R_{d_B}](\rho_{AB}) \geq 0\}$



Problem

Compare the convex sets

$$\mathcal{SEP} \subseteq \mathcal{PPT} \subseteq \mathcal{RED} \subseteq \mathcal{M}^{1,+}(\mathbb{C}^{d_A d_B}).$$

- For $(d_A, d_B) \in \{(2,2),(2,3),(3,2)\}$ we have $\mathcal{SEP} = \mathcal{PPT}$. In other dimensions, the inclusion $\mathcal{SEP} \subset \mathcal{PPT}$ is strict.
- For $d_B = 2$ we have $\mathcal{PPT} = \mathcal{RED}$. In other dimensions, the inclusion $\mathcal{PPT} \subset \mathcal{RED}$ is strict.

Random quantum states

Probability measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- We want to measure volumes of subsets of $\mathcal{M}_d^{1,+}(\mathbb{C})$, with $d = d_A d_B$.
- A natural choice is to use the Lebesgue measure (see $\mathcal{M}_d^{1,+}(\mathbb{C})$ as a compact subset of $\mathcal{M}_d^{sa}(\mathbb{C})$). The set of separable states $\mathcal{SEP}$ has positive volume, since $\mathcal{SEP}$ contains an open ball around I/d.
- Another choice - open quantum systems point of view: assume your system Hilbert space $\mathbb{C}^d = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ is coupled to an environment $\mathbb{C}^{d_C}$.
- On the tri-partite system $\mathcal{H}_{ABC} = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_C}$, consider a random pure state $|\psi\rangle_{ABC}$, i.e. a uniform random point on the unit sphere of the total Hilbert space $\mathcal{H}_{ABC}$.
- Trace out the environment $\mathbb{C}^{d_C}$ to get a *random density matrix*

$$\rho_{AB} = \mathrm{Tr}_C |\psi\rangle_{ABC}\langle\psi|.$$

- These probability measures have been introduced by Życzkowski and Sommers and they are called the induced measures of parameters $d = d_A d_B$ and $s = d_C$; we denote them by $\mu_{d,s}$.
- Remarkably, the Lebesgue measure is obtained for $s = d$.

Probability measures on $\mathcal{M}_d^{1,+}(\mathbb{C})$

- Here's an equivalent way of defining the measures $\mu_{d,s}$, in the spirit of Random Matrix Theory.
- Let $X \in \mathcal{M}_{d\times s}(\mathbb{C})$ be a $d \times s$ matrix with i.i.d. complex standard Gaussian entries (i.e. a Ginibre random matrix). Define

$$W_{d,s} = XX^* \text{ and } \mathcal{M}^{1,+}(\mathbb{C}^d) \ni \rho_{d,s} = \frac{XX^*}{\mathrm{Tr}(XX^*)} = \frac{W_{d,s}}{\mathrm{Tr}\, W_{d,s}}.$$

- The random matrix $W_{d,s}$ is called a Wishart matrix and the distribution of $\rho_{d,s}$ is precisely $\mu_{d,s}$.
- The measure $\mu_{d,s}$ is unitarily invariant: if $\rho \sim \mu_{d,s}$ and U is a fixed unitary matrix, then $U\rho U^* \sim \mu_{d,s}$.
- Density of $\mu_{d,s}$: $\mathrm{d}\mathbb{P}(\rho) = C_{d,s} \det(\rho)^{s-d} \mathbf{1}_{\rho \geq 0,\, \mathrm{Tr}\,\rho = 1}\, \mathrm{d}\rho$.
- Integrating out the eigenvectors, we obtain the eigenvalue density formula for random quantum states:

$$\mathrm{d}\mathbb{P}(\lambda_1, \ldots, \lambda_d) = C'_{d,s} \left[\prod_i \lambda_i^{s-d}\right] \left[\prod_{i<j} (\lambda_i - \lambda_j)^2\right] \mathbf{1}_{\lambda_i \geq 0,\, \sum_i \lambda_i = 1}\, \mathrm{d}\lambda.$$

Eigenvalues for induced measures

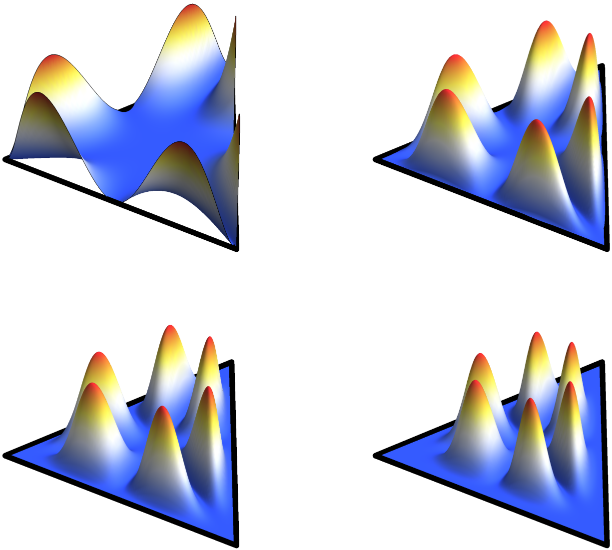


Figure: Induced measures for $d = 3$ and $s = 3, 5, 7, 10$.

Eigenvalues for induced measures

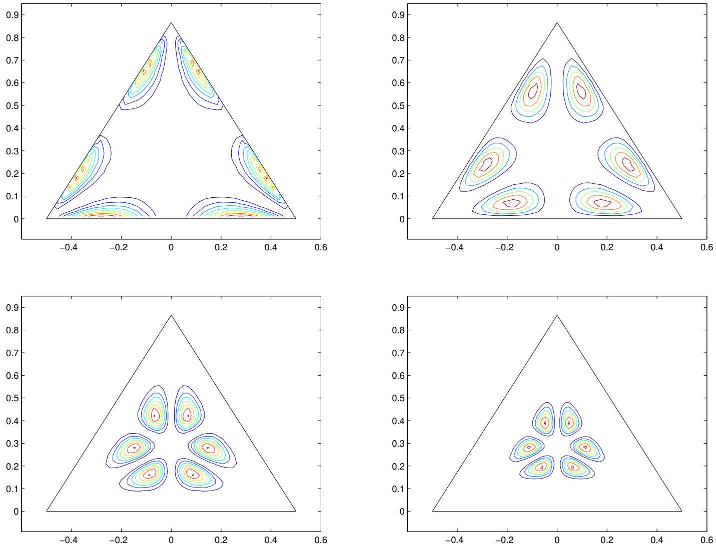


Figure: Induced measures for $d = 3$ and $s = 3, 5, 7, 10$.

Thresholds for entanglement criteria

Volume of convex sets under the induced measures

- Fix d, and let $C \subset \mathcal{M}^{1,+}(\mathbb{C}^d)$ a convex body, with $\mathrm{I}_d/d \in \mathrm{int}(C)$. Then

$$\lim_{s\to\infty} \mu_{d,s}(C) = 1.$$

In other words, the eigenvalues of a random density matrix $\rho_{AB} \sim \mu_{d,s}$ with d fixed and $s \to \infty$ converge to $1/d$.

Definition

A pair of functions $(s_0(d), s_1(d))$ are called a threshold for a family of convex sets $(K_d)_d$ if both conditions below hold

- If $s(d) \lesssim s_0(d)$, then

$$\lim_{d\to\infty} \mu_{d,s(d)}(K_d) = 0;$$

- If $s(d) \gtrsim s_1(d)$, then

$$\lim_{d\to\infty} \mu_{d,s(d)}(K_d) = 1.$$

Thresholds for entanglement criteria

- Below, the threshold functions $s_{0,1}(d)$ are of the form

$$s_0(d) = s_1(d) = cd; \qquad \text{we put } r := \min(d_A, d_B).$$

Crit. \ Reg.	$d_A = d_B \to \infty$	$d_B \to \infty$	$d_A \to \infty$
$\mathcal{SEP}$	$\infty \;\; (r \lesssim c \lesssim r \log^2 r)$	?	?
$\mathcal{PPT}$	4	$2 + 2\sqrt{1 - \frac{1}{r^2}}$	$2 + 2\sqrt{1 - \frac{1}{r^2}}$
$\mathcal{RED}$	0	0	$\frac{(1+\sqrt{r+1})^2}{r(r-1)}$

- The results in the table above can be interpreted in the following way: for a convex set K having a threshold c, a random density matrix $\rho_{AB} \sim \mu_{d,s}$ with large s, d will satisfy
  - If $s/d > c$, $\mathbb{P}[\rho_{AB} \in K] \approx 1$
  - If $s/d < c$, $\mathbb{P}[\rho_{AB} \in K] \approx 0$.

Proof elements

- The main task is to compute the probability that some random matrices are positive semidefinite or not.
- This is a very difficult computation to perform at fixed Hilbert space dimension; the asymptotic theory is much easier (one or both $d_{A,B} \to \infty$).
- To a selfadjoint matrix $X \in \mathcal{M}_d(\mathbb{C})$, with spectrum $x = (x_1, \ldots, x_d)$, associate its empirical spectral distribution
$$\mu_X = \frac{1}{d} \sum_{i=1}^{d} \delta_{x_i}.$$
- The probability measure μ_X contains all the information about the spectrum of X.
- A sequence of matrices X_d converges in moments towards a probability measure μ if, for all integer $p \geq 1$,
$$\lim_{d \to \infty} \frac{1}{d} \mathrm{Tr}(X_d^p) = \lim_{d \to \infty} \int x^p d\mu_{X_d}(x) = \int x^p d\mu(x).$$

Wishart matrices

Theorem (Marcenko-Pastur)

Let W be a complex Wishart matrix of parameters (d, cd). Then, almost surely with $d \to \infty$, the empirical spectral distribution of W/d converges in moments to a free Poisson distribution (a.k.a. Marčenko-Pastur distribution) π_c of parameter c.

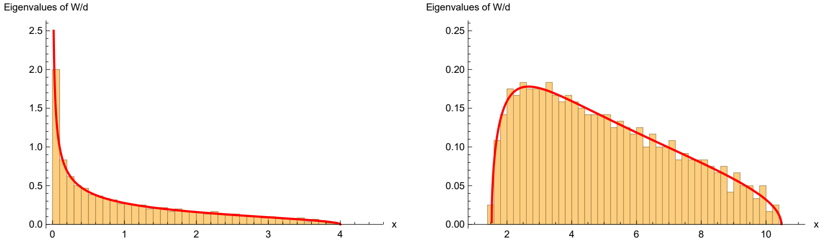


Figure: Eigenvalue distribution for Wishart matrices. In blue, the density of theoretical limiting distribution, π_c. In the two pictures, $d = 1000$, and $c = 1, 5$.

Partial transposition of a Wishart matrix

Theorem (Banica, N.)

Let W be a complex Wishart matrix of parameters (dn, cdn). Then, almost surely with $d \to \infty$, the empirical spectral distribution of $[\mathrm{id} \otimes \Theta](W_{AB}/d)$ converges in moments to a free difference of free Poisson distributions of respective parameters $cn(n \pm 1)/2$.

Corollary

The limiting measure above has positive support iff

$$c > c_{PPT} := 2 + 2\sqrt{1 - \frac{1}{n^2}}.$$

Partial transposition criterion - numerics

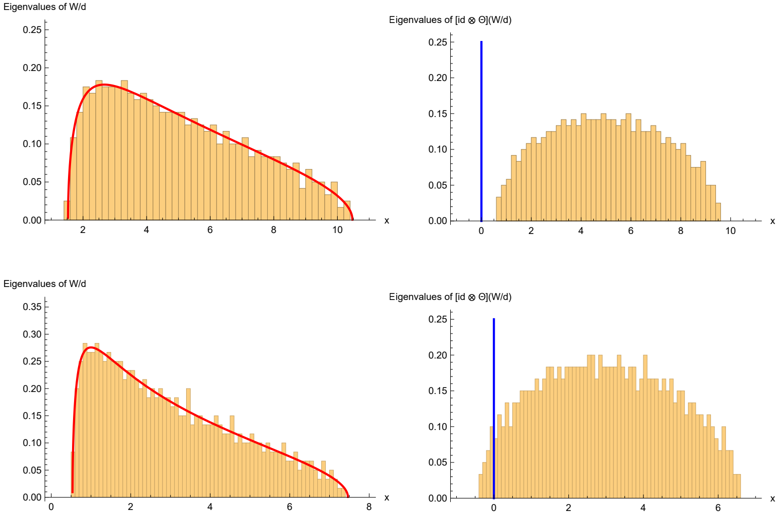


Figure: Wishart matrices before (left) and after (right) the application of the partial transposition. Here, $d = d_A = 200$, $n = d_B = 3$, and $c = 5$ (top), $c = 3$ (bottom). Note that $5 > c_{PPT} = 3.88562 > 3$.

Reduction of a Wishart matrix

Theorem (Jivulescu, Lupa, N.)

Let W be a complex Wishart matrix of parameters (dn, cdn). Then, almost surely with $d \to \infty$, the empirical spectral distribution of $[\mathrm{id} \otimes R](W_{AB}/d)$ converges in moments to a compound free Poisson distribution $\pi_{\nu_{n,c}}$ of parameter $\nu_{n,c} = c\delta_{1-n} + c(n^2-1)\delta_1$.

Corollary

The limiting measure above has positive support iff

$$c > c_{RED} := \frac{(1+\sqrt{n+1})^2}{n(n-1)}.$$

Remark

We have, for $n = 2$, $c_{PPT} = c_{RED} = 2 + \sqrt{3}$: the two criteria are know to be equivalent for qubit-qudit systems. For $n \geq 3$, we have $c_{PPT} > c_{RED}$: the reduction criterion is, in general, weaker than the PPT criterion.

Reduction criterion - numerics

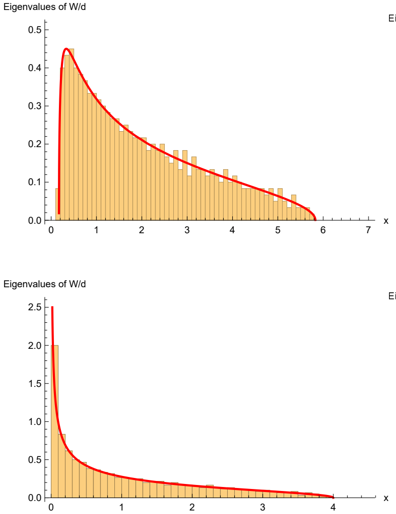


Figure: Wishart matrices before (left) and after (right) the application of the partial reduction map. Here, $d = d_A = 200$, $n = d_B = 3$, and $c = 2$ (top), $c = 1$ (bottom). Note that $2 > c_{RED} = 1.5 > 1$.

Random matrices and free probability

The free additive convolution of probability measures

- Given two self-adjoint matrices X, Y with spectra x, y, what is the spectrum of $X + Y$?
- In general, a very difficult problem, the answer depends on the relative position of the eigenspaces of X and Y (Horn problem).
- When the size of X, Y is large, and the eigenvectors are in general position, free probability theory gives the answer.
- Free additive convolution of two compactly supported probability distributions μ, ν: sample $x, y \in \mathbb{R}^d$ from μ, ν and consider

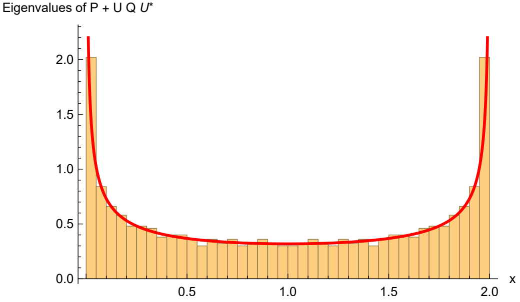
$$Z := \mathrm{diag}(x) + U\mathrm{diag}(y)U^*,$$

  where U is a $d \times d$ Haar unitary random matrix. Then, as $d \to \infty$, the empirical eigenvalue distribution of Z converges to a probability measure denoted by $\mu \boxplus \nu$.
- The operation $\boxplus$ is called free additive convolution, and it can be computed via the $\mathcal{R}$-transform (a kind of Fourier transform in the free world)

Free additive convolution - an example

- We have

$$\left[\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right] \boxplus \left[\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right] = \frac{1}{\pi\sqrt{x(2-x)}}\mathbf{1}_{(0,2)}(x)\,\mathrm{d}x.$$



- Compare to the classical situation, where $*$ denotes the (additive) classical convolution

$$\left[\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right] * \left[\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right] = \frac{1}{4}\delta_0 + \frac{1}{2}\delta_1 + \frac{1}{4}\delta_2.$$

The free Poisson distribution

- The limiting distribution of Wishart matrices (and of random density matrices from $\mu_{d,cd}$) is the free Poisson distribution

$$\pi_c := \max(1-c,0)\delta_0 + \frac{\sqrt{4c-(x-1-c)^2}}{2\pi x}\mathbf{1}_{[(1-\sqrt{c})^2,(1+\sqrt{c})^2]}(x)\,\mathrm{d}x.$$

- One can show a free Poisson Central Limit Theorem:

$$\lim_{n\to\infty}\left[\left(1-\frac{c}{n}\right)\delta_0+\frac{c}{n}\delta_1\right]^{\boxplus n}=\pi_c.$$

- The limit measure for $[\mathrm{id}\otimes\Theta](W_{AB}/d)$ is

$$\pi_c^{PPT} := \pi_{cn(n+1)/2}\boxplus D_{-1}\pi_{cn(n-1)/2}.$$

- The free compound Poisson measure of parameter ν is defined via a generalized free Poisson central limit theorem

$$\lim_{n\to\infty}\left[\left(1-\frac{\nu(\mathbb{R})}{n}\right)\delta_0+\frac{1}{n}\nu\right]^{\boxplus n}=:\pi_\nu.$$

- The limit measure for $[\mathrm{id}\otimes R](W_{AB}/d)$ is

$$\pi_c^{RED} := \pi_{c\delta_{1-n}+c(n^2-1)\delta_1}.$$

Thank you!

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