Additivité de la capacité des canaux quantiques

comment l'intrication peut aider la transmission de l'information

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Rencontre LPT-IMT
Toulouse, 4 Avril 2011

Random quantum channels
&
additivity problems

Additivity for MOE of quantum channels

- Quantum channels: CPTP maps $\Phi : \mathcal{M}_{\text{in}}(\mathbb{C}) \to \mathcal{M}_{\text{out}}(\mathbb{C})$
  1. Completely Positive: $\Phi \otimes \text{id}_k \geqslant 0, \forall k$.
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- NO !!!
  - $p > 1$: Hayden '07, Hayden, Winter '08;
  - $p = 1$: Hastings '08, Aubrun, Szarek, Werner '10

Importance of additivity

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- Difficult, mathematically challenging problem.

Random quantum channels

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$$\Phi(\rho) = \mathrm{Tr}_{\mathrm{aux}}(V \rho V^*),$$

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- Equivalently, via the Stinespring dilation theorem

$$\Phi(\rho) = \mathrm{Tr}_{\mathrm{aux}}(U(\rho \otimes P_y)U^*),$$

where $y \in \mathbb{C}^{\frac{\mathrm{out} \times \mathrm{aux}}{\mathrm{in}}}$ and $U \in \mathcal{M}_{\mathrm{out} \times \mathrm{aux}}(\mathbb{C})$ is a Haar unitary matrix.

Models of interest

Finite rank output

- $\text{in} = tnk$,
- $\text{out} = k$,
- $\text{aux} = n$,

where $n, k \in \mathbb{N}$ and $t \in (0, 1)$. In general, we shall assume that

- $n \to \infty$ and k is fixed, but "large";
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Unbounded rank output

- $\text{in} = n$,
- $\text{out} = n$,
- $\text{aux} = k$,

where $n, k \in \mathbb{N}$ such that

- $n, k \to \infty$;
- $k/n \to c$, where $c > 0$ is a constant parameter.

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Strategy for the product

- Use trivial bound

$$H^p_{\min}(\Phi \otimes \overline{\Phi}) \leqslant H^p\left([\Phi \otimes \overline{\Phi}](X_{12})\right),$$

for a particular choice of $X_{12} \in \mathcal{M}_{tnk}(\mathbb{C}) \otimes \mathcal{M}_{tnk}(\mathbb{C})$.

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- Bound entropies of the (random) density matrix

$$Z = [\Phi \otimes \overline{\Phi}](E_{\text{in}}) \in \mathcal{M}_{\text{in}}(\mathbb{C}) \otimes \mathcal{M}_{\text{in}}(\mathbb{C}).$$

Main result - 1 channel, finite rank output

Theorem (Belinschi, Collins, N. '10)

For a quantum channel $\Phi : \mathcal{M}_{tnk}(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$ *the set of possible output states converges, in the limit* $n \to \infty$, k, t *fixed, to a deterministic limit set. The set of possible output eigenvalues is a convex subset* $K_{k,t}$ *of the probability simplex, described by free probability.*

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- It is not just a bound, the exact limiting value is obtained. In other words, the problem is solved for 1 channel !
- However, the set of possible output states is not explicit, and minimizing entropy functions is mathematically difficult (work in progress).

Main result - 2 channels, finite rank output

Theorem (Collins, N. '09)

For all k, t, almost surely as $n \to \infty$, the eigenvalues of $Z = [\Phi \otimes \overline{\Phi}](E_{tnk})$ converge to

$$\left(t + \frac{1-t}{k^2}, \underbrace{\frac{1-t}{k^2}, \ldots, \frac{1-t}{k^2}}_{k^2-1 \text{ times}} \right).$$

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- Two improvements:
  1. "better" largest eigenvalue,
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- However, smaller eigenvalues are the "worst possible".

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Theorem (Collins, N. '09)

For all $c > 0$, the eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_{n^2}$ of $Z = [\Phi \otimes \overline{\Phi}](E_n)$ satisfy:

- *In probability, $cn\lambda_1 \to 1$.*
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Free Poisson distribution

- The free Poisson distribution of parameter $c > 0$ is given by

$$\pi_c = \max(1-c, 0)\delta_0 + \frac{\sqrt{4c - (x-1-c)^2}}{2\pi x} \mathbf{1}_{[1+c-2\sqrt{c}, 1+c+2\sqrt{c}]}(x) \, dx.$$

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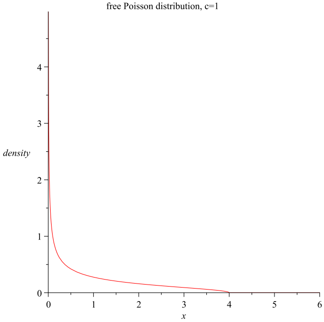

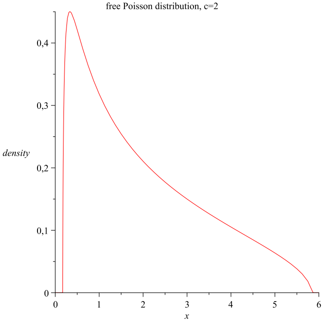

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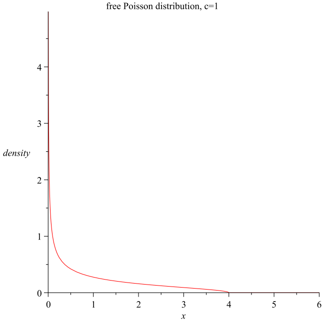

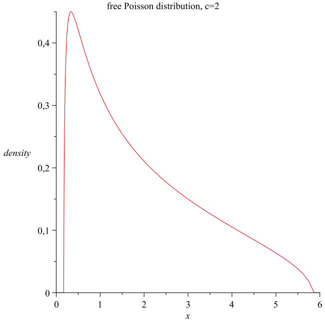

- Free Poisson Central Limit Theorem:

$$\left[\left(1 - \frac{c}{n}\right)\delta_0 + \frac{c}{n}\delta_1\right]^{\boxplus n} \to \pi_c.$$

Independent channels - unbounded rank

Theorem (Collins + N. '09)

For $c > 0$, consider two independent random quantum channels Φ and Ψ. The eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_{n^2}$ of $Z = [\Phi \otimes \Psi](E_n)$ are such that almost surely,

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Graphical calculus for random quantum channels

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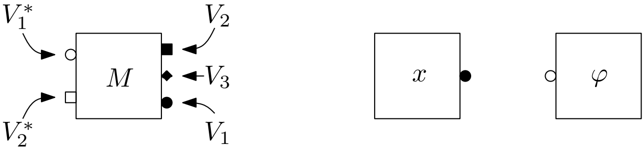

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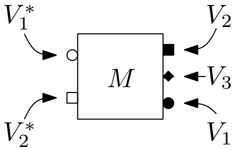

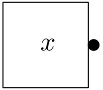

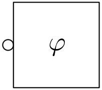

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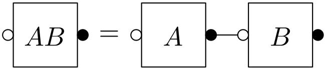
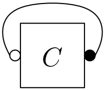
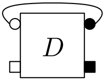




$\mathrm{Tr(C)}$



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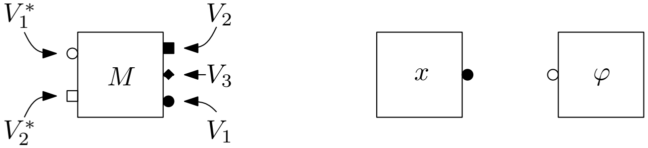

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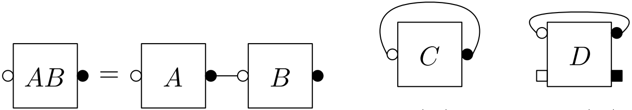
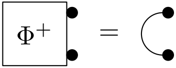


$\mathrm{Tr(C)}$ $\quad \mathrm{Tr_{V_1}(D)}$

- Bell state $\Phi^+ = \sum_{i=1}^{\dim V_1} e_i \otimes e_i \in V_1 \otimes V_1$



Graphical representation of quantum channels

- Decorations/labels

●/○ $= \mathbf{C}^n$ ■/□ $= \mathbf{C}^k$ ◆/◇ $= \mathbf{C}^{tnk}$ ▲/△ $= \mathbf{C}^{t^{-1}}$

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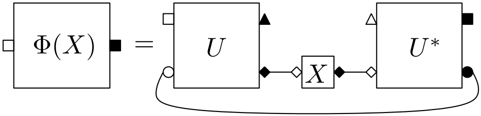

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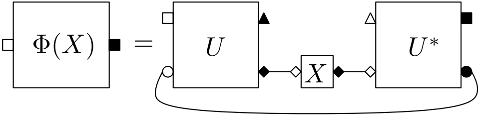

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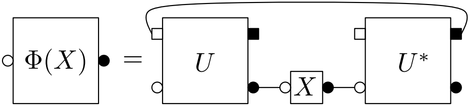

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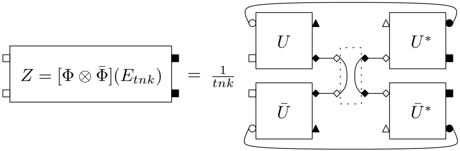

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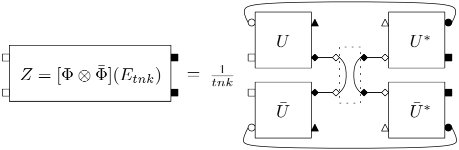

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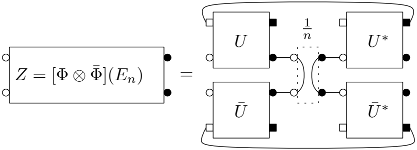

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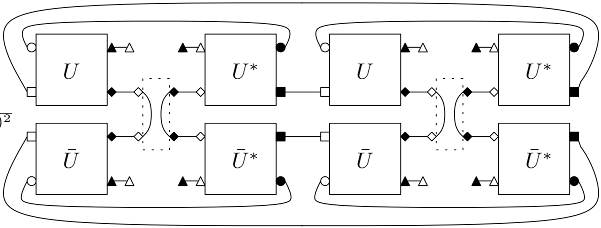

Unitary integration - Weingarten formula

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- There is a graphical way of reading this formula on the diagrams !

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5. Erase all U and $\overline{U}$ boxes. The resulting diagram is denoted by $\mathcal{D}_{(\alpha,\beta)}$.

Theorem

$$\mathbb{E}\mathcal{D} = \sum_{\alpha,\beta} \mathcal{D}_{(\alpha,\beta)} \operatorname{Wg}(d, \alpha\beta^{-1}).$$

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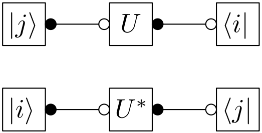

Figure: Diagram for $|u_{ij}|^2 = U_{ij} \cdot (U^*)_{ji}$.

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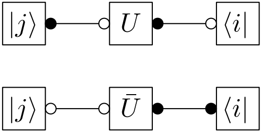

Figure: The U^* box replaced by an $\bar{U}$ box.

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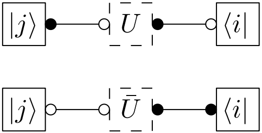

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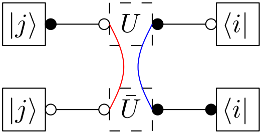

Figure: Pair white decorations (red wires) and black decorations (blue wires); only one possible pairing : $\alpha = (1)$ and $\beta = (1)$.

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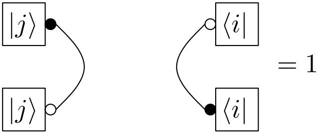

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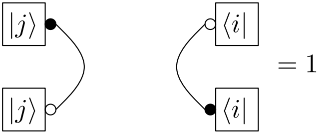

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- Conclusion :
$\mathbb{E}|u_{ij}|^2 = \int |u_{ij}|^2 \, \mathrm{d}U = \mathcal{D}_{\alpha=(1),\beta=(1)} \cdot \mathrm{Wg}(N,(1)) = 1 \cdot 1/N = 1/N$.

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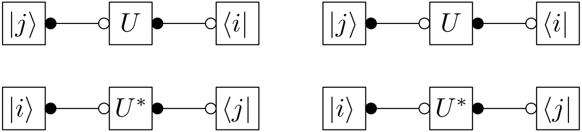

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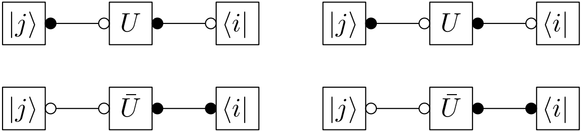

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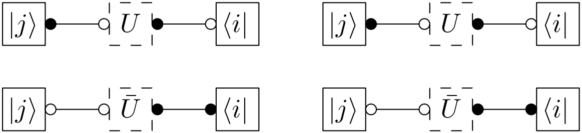

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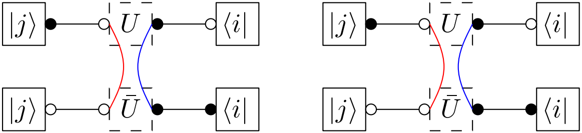

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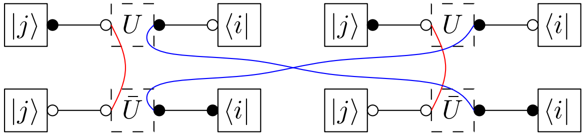

Figure: Second pairing : $\alpha = (1)(2)$ and $\beta = (12)$.

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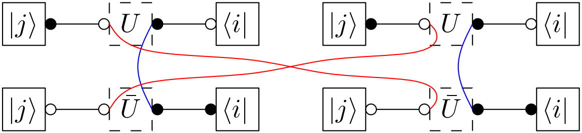

Figure: Third pairing : $\alpha = (12)$ and $\beta = (1)(2)$.

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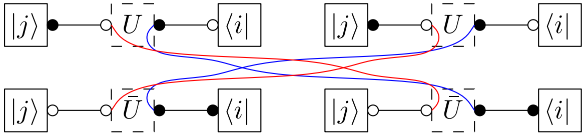

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- Conclusion :

$$\mathbb{E}|u_{ij}|^4 = \int |u_{ij}|^4 \,\mathrm{d}U =$$

$$\begin{aligned}
&\mathcal{D}_{(1)(2),(1)(2)} \cdot \mathrm{Wg}(N,(1)(2)) + \\
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&\mathcal{D}_{(12),(1)(2)} \cdot \mathrm{Wg}(N,(12)) + \\
&\mathcal{D}_{(12),(12)} \cdot \mathrm{Wg}(N,(1)(2)) \\
&= \mathrm{Wg}(N,(1)(2)) + \mathrm{Wg}(N,(12)) + \mathrm{Wg}(N,(12)) + \mathrm{Wg}(N,(1)(2)) \\
&= \frac{2}{N^2-1} - \frac{2}{N(N^2-1)} = \frac{2}{N(N+1)}.
\end{aligned}$$

Third example : twirling

- Consider a fixed matrix $A \in \mathcal{M}_N(\mathbb{C})$. Compute $\int_{\mathcal{U}(N)} U^* A U \, \mathrm{d}U$.

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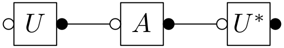

Figure: Diagram for $U^* A U$.

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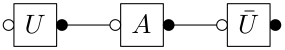

Figure: The U^* box replaced by an $\bar{U}$ box.

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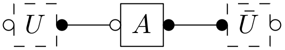

Figure: Erase U and $\bar{U}$ boxes.

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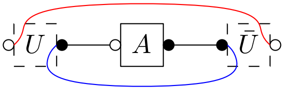

Figure: Pair white decorations (red wires) and black decorations (blue wires); only one possible pairing : $\alpha = (1)$ and $\beta = (1)$.

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- Conclusion : $\int_{\mathcal{U}(N)} U^* A U \, \mathrm{d}U = \mathcal{D}_{\alpha=(1),\beta=(1)} \cdot \mathrm{Wg}(N, (1)) = \frac{\mathrm{Tr}(A)}{N} \mathrm{I}_N$.

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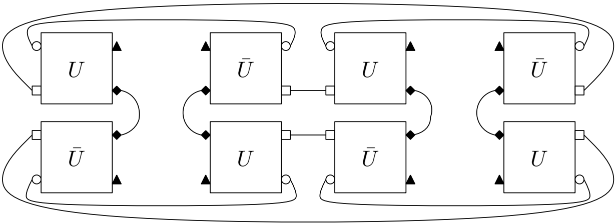

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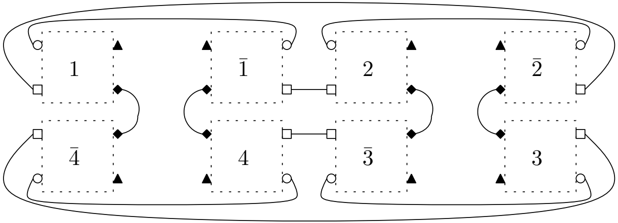
The diagram with the boxes removed



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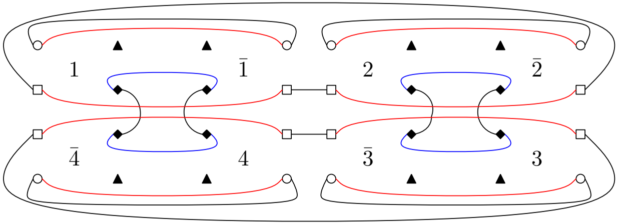

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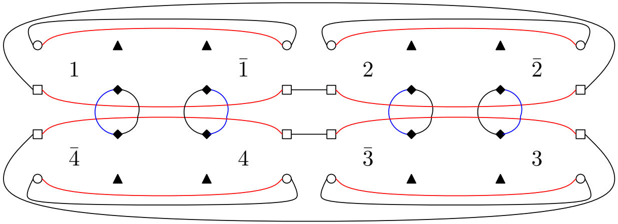

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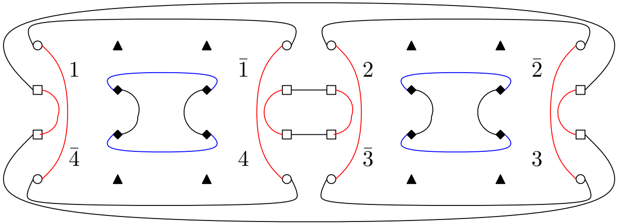

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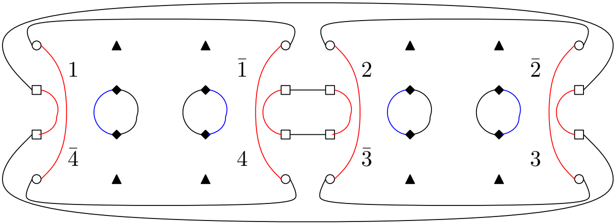

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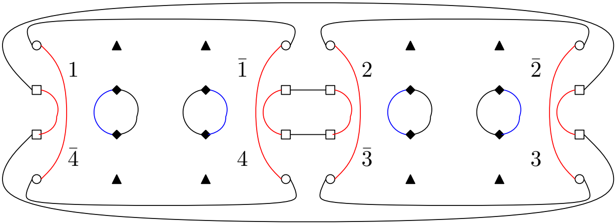

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- Contributions of diagrams $\rightsquigarrow$ counting the loops $\rightsquigarrow$ statistics over permutations.

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- The unbounded rank case for conjugate channels is more delicate, since the $n^2 - 1$ smaller eigenvalues are one order of magnitude below the largest eigenvalue. One needs to consider the eigenspace compression QZQ, where $Q = \mathrm{I} - E_n$.

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- Geodesic problems in symmetric groups $\Rightarrow$ non-crossing partitions $\Rightarrow$ free probability.
- The free Poisson distribution is characterized by its moments:

$$\int x^p \, d\pi_c(x) = \sum_{\substack{\alpha \in \mathcal{S}_p \\ \#\alpha + \#(\gamma^{-1}\alpha) = p+1}} c^{\#\alpha}.$$

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- Other applications to QIT (work in progress with B. Collins and K. Życzkowski)

Merci !