On the number of components of random meandric systems

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I Number of components of meandric systems

Notation

- $NC(n) :=$ set of all non-crossing partitions of $\{1, \dots, n\}$

- $NC_2(2n) :=$ set of all non-crossing pair-partitions of $\{1, \dots, 2n\}$

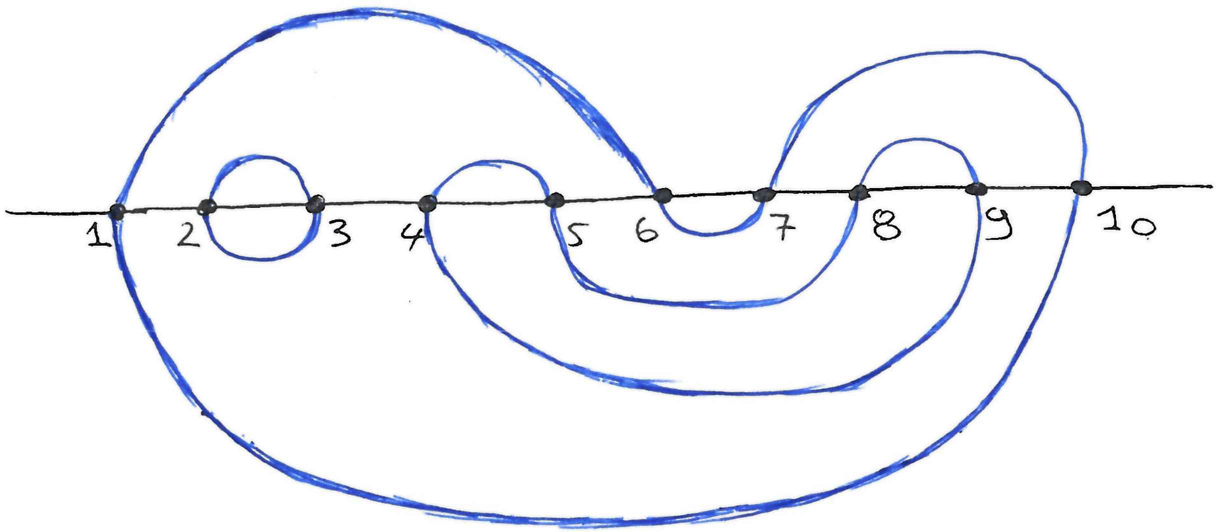
- Have $|NC(n)| = |NC_2(2n)| = Cat_n = \frac{(2n)!}{n!\,(n+1)!}$

  (n-th Catalan number)

- For every $(\sigma, \tau) \in NC_2(2n)^2$ we can draw a meandric system, denoted as $M_{\sigma,\tau}$

$(\sigma,\tau) \in NC_2(2n)^2 \rightsquigarrow$ meandric system $\underline{M_{\sigma,\tau}}$

Denote $\underline{\#(M_{\sigma,\tau})}$:= number of connected components of $M_{\sigma,\tau}$



[A meandric system on 10 points, which has 3 connected components]

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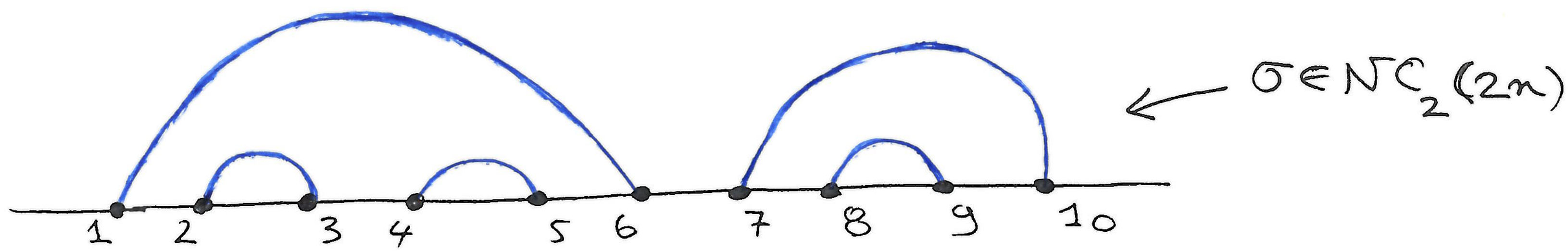

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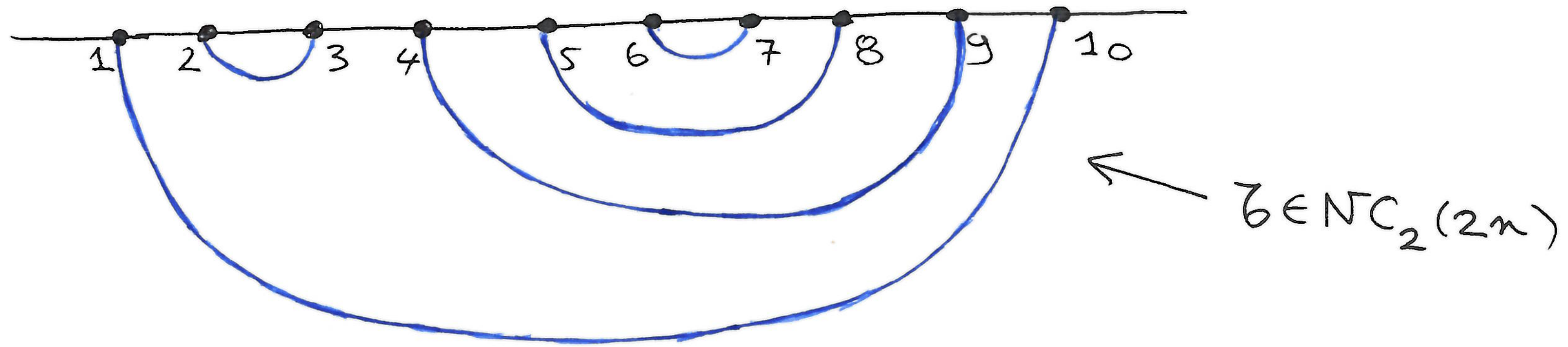

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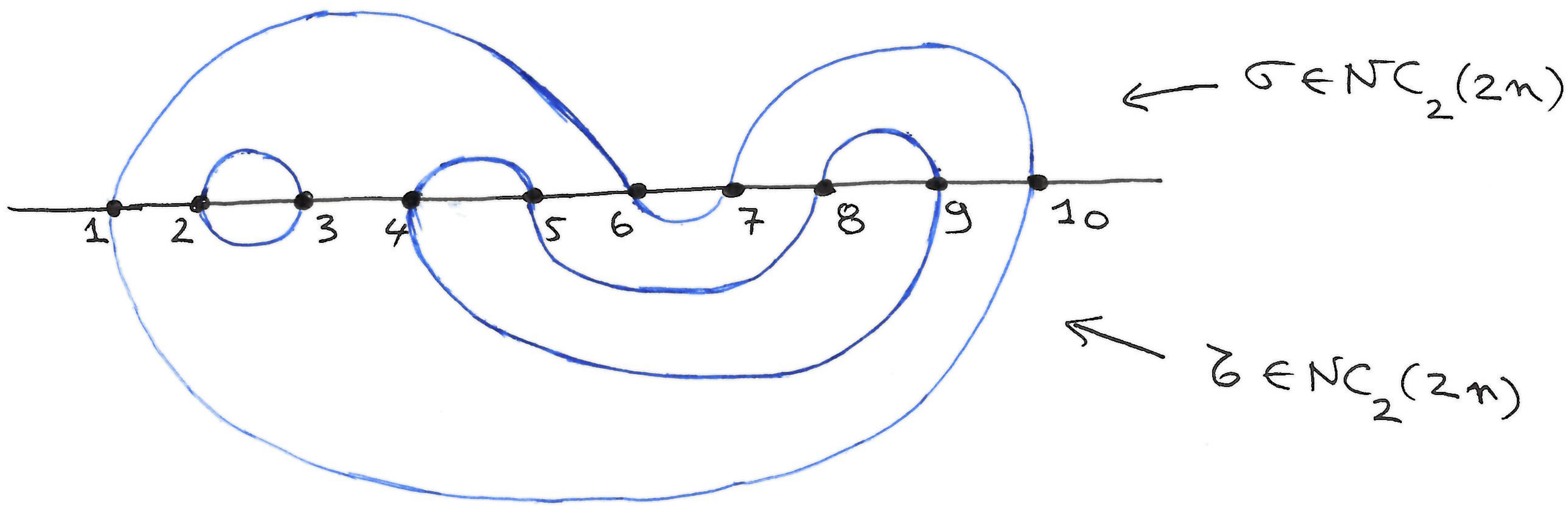


[A meandric system on 10 points, which has 3 connected components]

An interesting sequence of random variables:

$$X_n : NC_2(2n)^2 \to \{1,\dots,n\}, \quad \boxed{X_n(\pi,\rho) = \#(M_{\pi,\rho})}$$

Have old open problem about

$$P(X_n = 1) = \frac{|\{(\sigma,\tau)\in NC_2(2n)^2 \mid \#(M_{\sigma,\tau}) = 1\}|}{Cat_n^2},$$

asking:

$$\underbrace{\lim \left[P(X_n = 1) \right]^{1/n} = ?}$$

limit believed to exist

numerics give $\approx \frac{12.26}{16}$

Much more basic (but also open): $\underline{E(X_n) = ?}$

(Obvious: $1 \leq E(X_n) \leq n, \quad \forall n \in \mathbb{N}$.)

Conjecture: $\lim\limits_{n \to \infty} \dfrac{E(X_n)}{n}$ exists, and is $\underbrace{\approx 0.23}$

Simulations

Can prove:

Theorem 1 (Goulden-Nica-Puder, arXiv:1708.05188)

$$\begin{cases} \liminf\limits_{n \to \infty} \dfrac{E(X_n)}{n} \geq 0.17 \\ \\ \limsup\limits_{n \to \infty} \dfrac{E(X_n)}{n} \leq 0.5 \end{cases}$$

II Equivalent framework: Hasse diagram of $NC(n)$

This is an undirected graph associated to the partial order by reverse refinement on $NC(n)$.

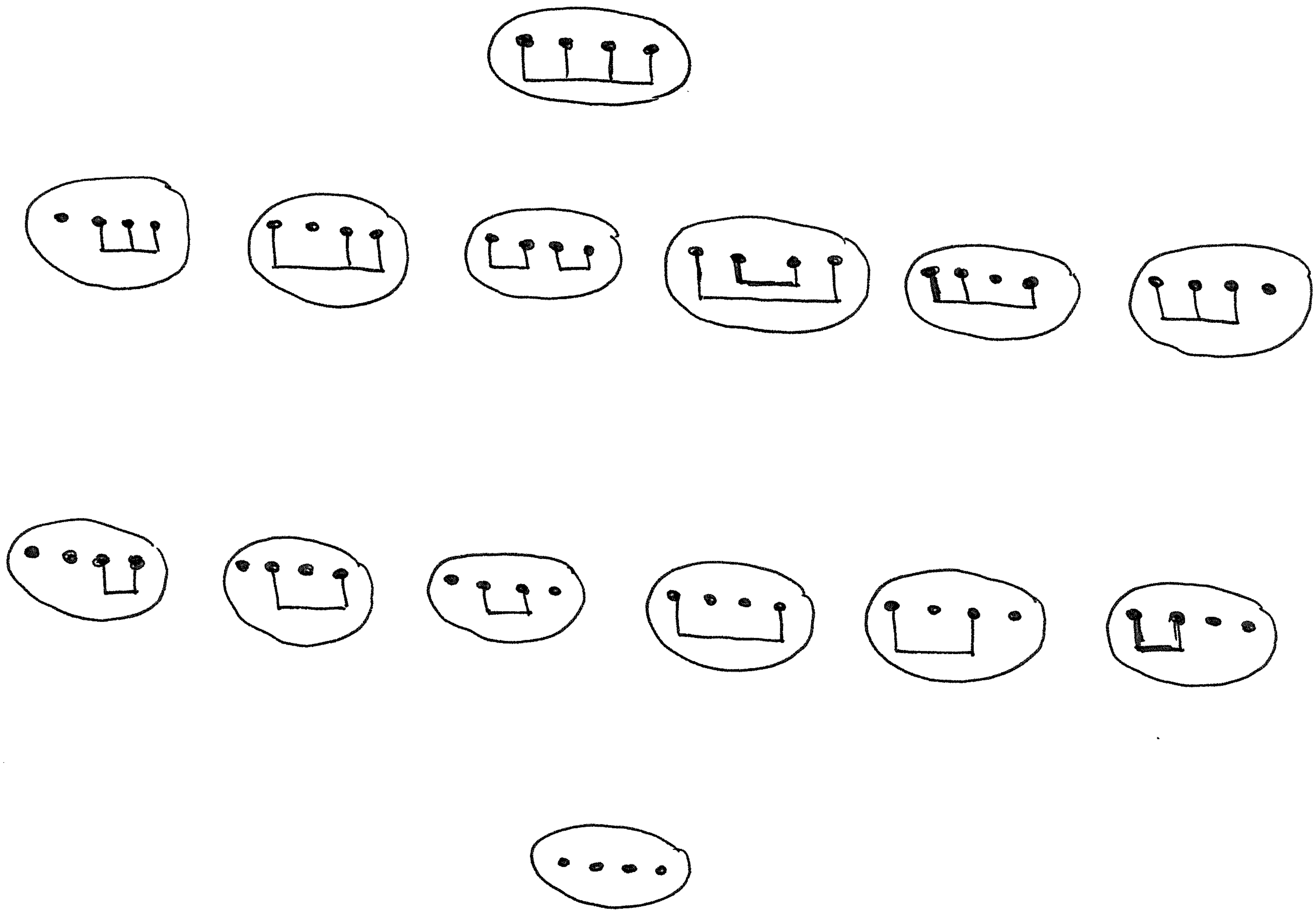
Vertices of graph $\rightsquigarrow NC(n)$

Edges of graph? $\rightsquigarrow$ show on an example

For $\pi, \rho \in NC(n)$, will denote

$d_H(\pi, \rho) :=$ geodesic distance between π and ρ in the Hasse diagram of $NC(n)$.

Example: the Hasse diagram of NC(4)



1.3b

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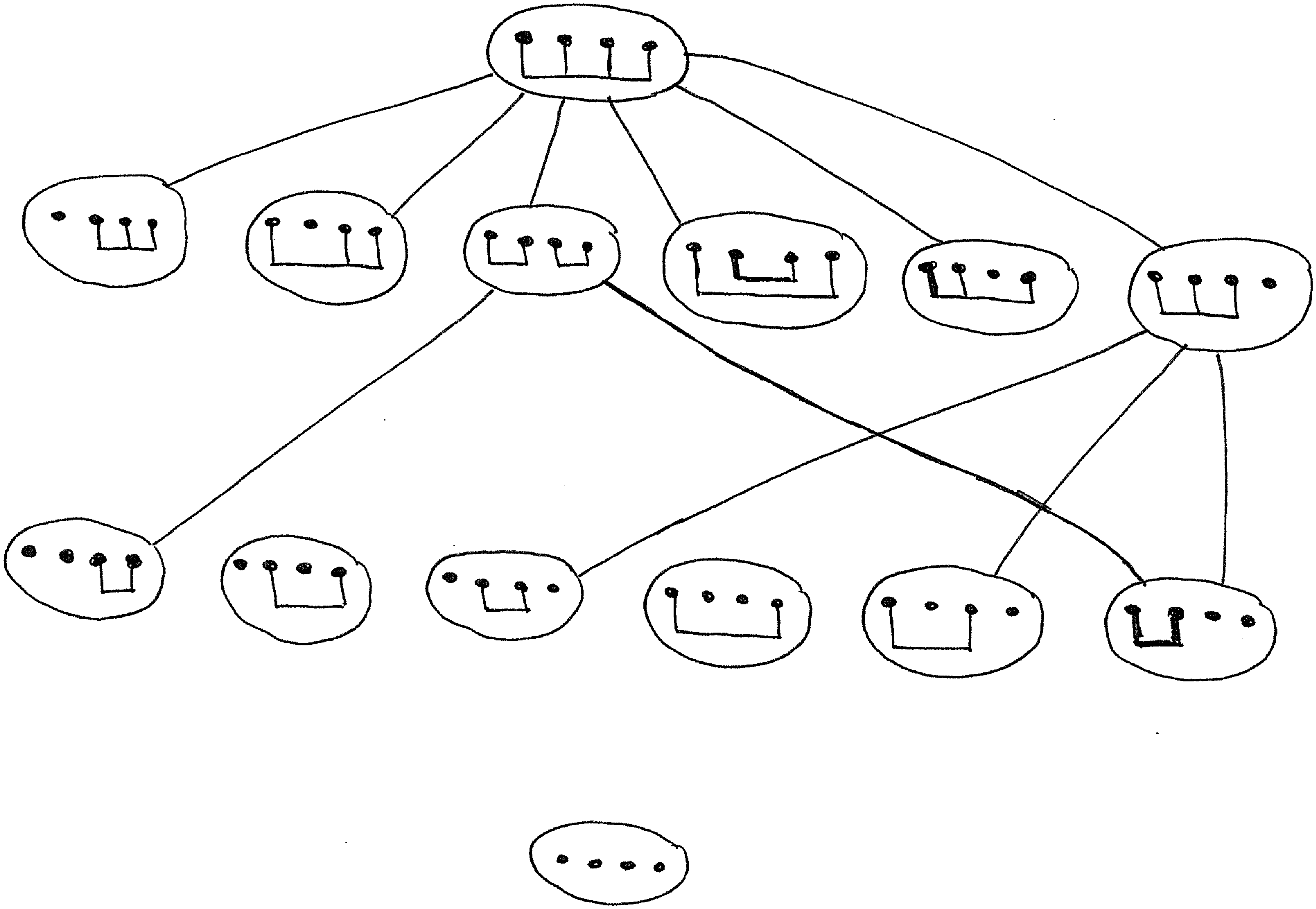

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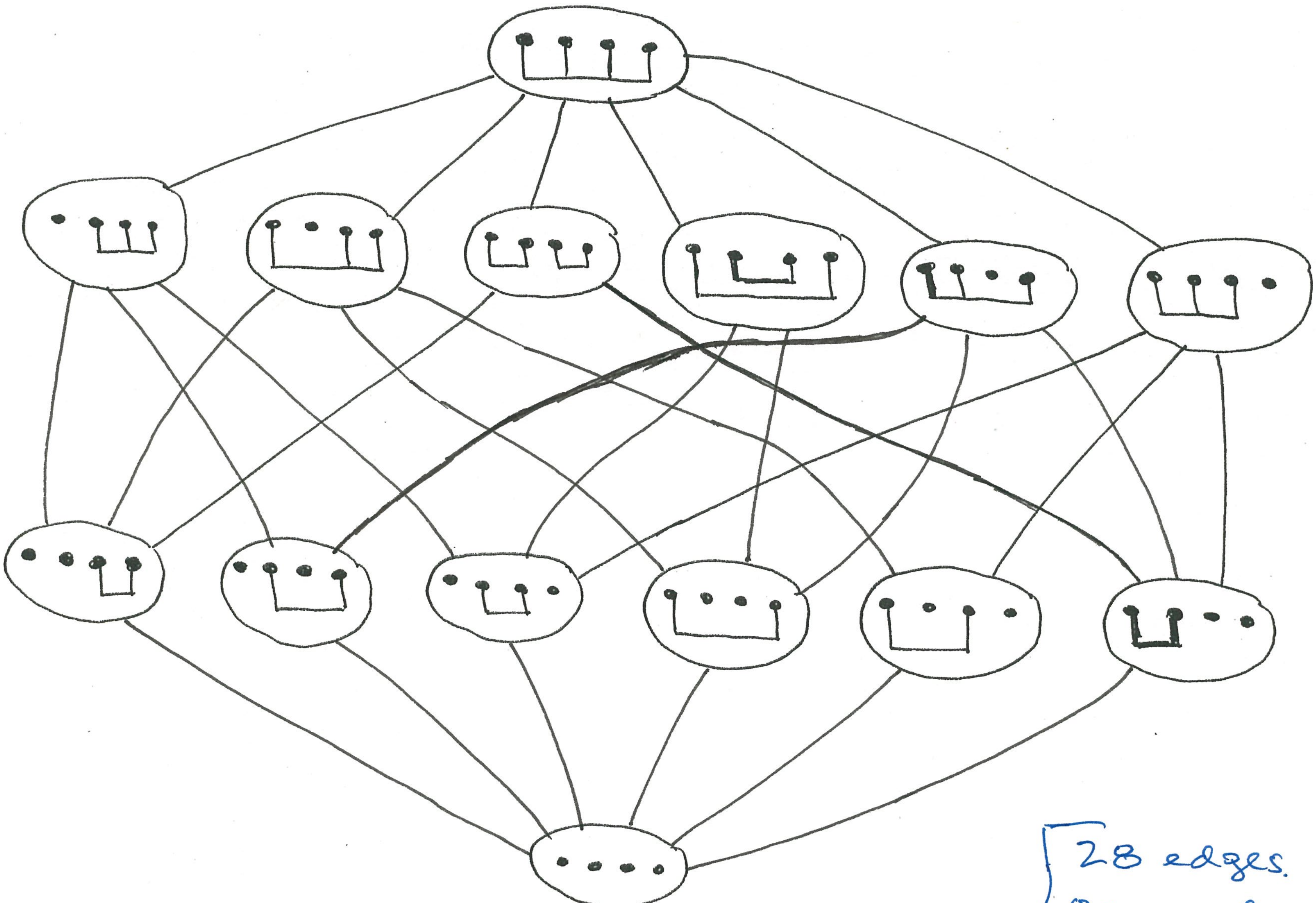
1.3c



28 edges. For general n, there are $\binom{2n}{n-2}$ edges.

Facts: **(1)** One has a natural bijection

$$NC(n) \ni \pi \mapsto \sigma \in NC_2(2n),$$

where σ is called the "fattening" of π

(2) [Result of Hall, Savitt, 2006.]

Let $\pi, \rho \in NC(n)$ and let $\tau, \sigma \in NC_2(2n)$ be the fattenings of π and ρ. Then

$$\boxed{\#(M_{\sigma,\tau}) = n - d_H(\pi, \rho)}$$

Consequence: The r.v. X_n from part I has

$$E(X_n) = n - E(Y_n),$$

where $\begin{cases} Y_n : NC(n)^2 \to \{0, 1, \ldots, n-1\} \\ Y_n(\pi, \rho) = d_H(\pi, \rho) \end{cases}$

$$\boxed{E(X_n) = n - E(Y_n), \quad \forall n \in \mathbb{N}.}$$

The conjecture about $E(X_n)$ then becomes:

Conjecture': $\lim_{n\to\infty} \frac{E(Y_n)}{n}$ exists and is $\simeq 0.77$

The related Theorem 1 becomes:

Theorem 1'

$$\begin{cases} \liminf_{n\to\infty} \frac{E(Y_n)}{n} \geq 0.5 \\ \\ \limsup_{n\to\infty} \frac{E(Y_n)}{n} \leq 0.83 \end{cases}$$

Explain the constants in Theorem 1'

$\boxed{\liminf_{n\to\infty} \frac{E(Y_n)}{n} \geq 0.5}$ follows easily from the

top-down symmetry of the Hasse diagram:

for every $n \in \mathbb{N}$ and $\pi, \rho \in NC(n)$, one has

$$d_H(\pi,\rho) + d_H(\pi, \underline{\rho^c}) \geq d_H(\rho,\rho^c) = n-1$$

Kreweras complement of ρ

by direct analysis

Sum over π, ρ to get $\underline{E(Y_n) \geq \frac{n-1}{2}, \ \forall n \in \mathbb{N}.}$

For $\boxed{\limsup_{n\to\infty} \frac{E(Y_n)}{n} \leq 0.83}$ we use a lucky reduction, via the inequality

$$d_H(\pi,\rho) \leq |\pi| + |\rho| - 2|\pi \vee \rho| \quad \Big(= d_H(\pi, \pi\vee\rho) + d_H(\pi\vee\rho, \rho)\Big)$$

holds with equality when one of π, ρ is an interval-partition.

Hence $E(Y_n) \leq E(|\pi|) + E(|\rho|) - 2E(|\pi\vee\rho|)$

$$= \frac{n+1}{2} + \frac{n+1}{2} - 2E(Z_n),$$

for $\begin{cases} Z_n : NC(n)^2 \longrightarrow \{1, \dots, n\} \\ Z_n(\pi,\rho) = |\pi \vee \rho| \end{cases}$

$$E(Y_n) \leq n+1-2E(Z_n)$$

$$\Downarrow$$

$$\limsup_{n\to\infty} \frac{E(Y_n)}{n} \leq 1 - 2 \lim_{n\to\infty} \frac{E(Z_n)}{n}$$

$$= 1 - 2 \cdot \frac{16-5\pi}{16-4\pi} = \frac{3\pi-8}{8-2\pi} < 0.83$$

by "Proposition on $E(Z_n)$" written on blackboard

Here $\pi = 3.141592\ldots$

III Approach to "Proposition on $E(Z_n)$" via $\boxplus$-powers

<u>Lemma 1.</u> $\mu : \mathbb{C}[X] \to \mathbb{C}$ linear, with $\mu(1)=1$.

Let $$U(t,z) := \sum_{n=1}^{\infty} \Big(\underbrace{\mu^{\boxplus t}(X^n)}_{\substack{\text{polynomial} \\ \text{in } t}} \Big) z^n \in \mathbb{C}[t][[z]].$$

Then U satisfies

$$\boxed{t \cdot \frac{\partial U}{\partial t} = U + \frac{z \cdot U \cdot \frac{\partial U}{\partial z}}{1+U}}$$

<u>Proof</u> Take $\frac{\partial}{\partial t}, \frac{\partial}{\partial z}$ in the functional equation for the R-transform of $\mu^{\boxplus t}$, then do suitable algebra. $\square$

<u>Remark</u> When in Lemma 1 we evaluate

U, $\frac{\partial U}{\partial t}$, $\frac{\partial U}{\partial z}$ at $t=1$, we get:

$$U(1,z) = M_\mu(z) = \sum_{n=1}^{\infty} \mu(x^n) z^n, \quad \frac{\partial U}{\partial z}(1,z) = M_\mu'(z),$$

and

$$\boxed{(*) \qquad \left.\frac{\partial U}{\partial t}\right|_{t=1}(z) = M_\mu(z) + \frac{z M_\mu(z) M_\mu'(z)}{1+M_\mu(z)}}$$

We will use $(*)$ for $\mu : \mathbb{C}[X] \to \mathbb{C}$.
defined by asking that $\mu(X^n) = \mathrm{Cat}_n^2$, $\forall n \in \mathbb{N}$.

Lemma 2. $\mu : \mathbb{C}[X] \to \mathbb{C}$ with $\mu(X^n) = \mathrm{Cat}_n^2$, $\forall n \in \mathbb{N}$.

Then for every $t > 0$ and $n \in \mathbb{N}$ we have

$$\mu^{\boxplus t}(X^n) = \sum_{\pi, \sigma \in NC(n)} t^{|\pi \vee \sigma|}$$

Proof. Follows from the explicit description of the free cumulants of μ given by Biane-Dehornoy 2014. $\square$

For μ in Lemma 2 we get $U(t,z) = \sum_{n=1}^{\infty} \Big(\sum_{\pi, \sigma \in NC(n)} t^{|\pi \vee \sigma|} \Big) z^n$,

hence

$$(**) \quad \frac{\partial U}{\partial t}\Big|_{t=1}(z) = \sum_{n=1}^{\infty} \Big(\underbrace{\sum_{\pi, \sigma \in NC(n)} |\pi \vee \sigma|}_{\text{This is } \mathrm{Cat}_n^2 \cdot E(Z_n)} \Big) z^n$$

For μ of Lemma 2 we obtained:

$$\boxed{(*)\quad \frac{\partial U}{\partial t}\Big|_{t=1}(z) = M_\mu(z) + \frac{z M_\mu(z) M_\mu'(z)}{1+M_\mu(z)}}$$

$$= f(z) + \frac{z f(z) f'(z)}{1+f(z)} \qquad \text{for } f(z) = \sum_{n=1}^{\infty} Cat_n^2 z^n$$

asymptotics for coefficients can be calculated explicitly

$$\boxed{(**)\quad \frac{\partial U}{\partial t}\Big|_{t=1}(z) = \sum_{n=1}^{\infty} \left(Cat_n^2 \cdot E(Z_n) \right) z^n.}$$

When comparing the right-hand sides of $(*)$ and $(**)$, one gets a formula for $E(Z_n)$ which leads to the stated limit for $\frac{E(Z_n)}{n}$.

The random variable X_n

Recall that X_n is the number of components of a random meandric system on $2n$ points.

$$\mathbb{P}(X_n = 1) = \frac{\#\text{meanders}}{\text{Cat}_n^2} \qquad \text{(hard)}$$

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$$\mathbb{P}(X_n = n - r) = \quad ? \qquad r \text{ fixed}$$

$$\mathbb{P}(X_n = n) = \frac{\mathrm{Cat}_n}{\mathrm{Cat}_n^2} \qquad (\text{easy}, \pi = \rho)$$

Generating series

- Define

$$M_{n,r} := \{(\pi,\rho) \in NC(n) \,:\, d_H(\pi,\rho) = r\}$$

$$M(X,Y) := \sum_{n\geq 1}\sum_{r\geq 0} X^n Y^r |M_{n,r}|$$

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$$M(X, Y) = \sum_{n \geq 1} X^n \sum_{\pi, \rho \in NC(n)} Y^{d_H(\pi, \rho)}$$

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\end{aligned}$$

- $\rightsquigarrow$ recognize moment - free cumulant formula

Moment - free cumulant transformations

- Put

$$K_{n,r} := \{(\pi, \rho) \in NC(n) \,:\, \pi \vee \rho = 1_n, d_H(\pi, \rho) = r\}$$

$$K(X, Y) := \sum_{n \geq 1} \sum_{r \geq 0} X^n Y^r |K_{n,r}|.$$

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- The series M and K are related by the moment - free cumulant formula

$$M(X, Y) = K(X(1 + M(X, Y)), Y).$$

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- Using a similar reduction and a Kreweras complement, we can go deeper: if

$$I_{n,r} := \{(\pi, \rho) \in NC(n) \, : \, \pi \wedge \rho = 0_n, \pi \vee \rho = 1_n, d_H(\pi, \rho) = r\}$$

$$I(X, Y) := \sum_{n \geq 1} \sum_{r \geq 0} X^n Y^r |I_{n,r}|$$

then

$$K(X, Y) = I(X(1 + K(X, Y)), Y).$$

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- Recall

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and let M, K, I the respective generating series.

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- If $\mathcal{F}$ is the operation transforming free cumulant generating series into moment generating series, we conclude

$$\boxed{I \xmapsto{\mathcal{F}_X} K \xmapsto{\mathcal{F}_X} M}$$

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- Morally, the sets $I_{n,r}$ should be easier to enumerate...

Key technical lemma

Lemma

For fixed r, the series I has finite support in n. More precisely, $I_{n,r} = \emptyset$, unless $r + 1 \leq n \leq 2r + \mathbf{1}_{r=0}$.

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For $r = 1$, $I_{2,1} = \{($  $,$  $), ($  $,$  $)\}$, and all the other $I_{n,1}$ are empty.

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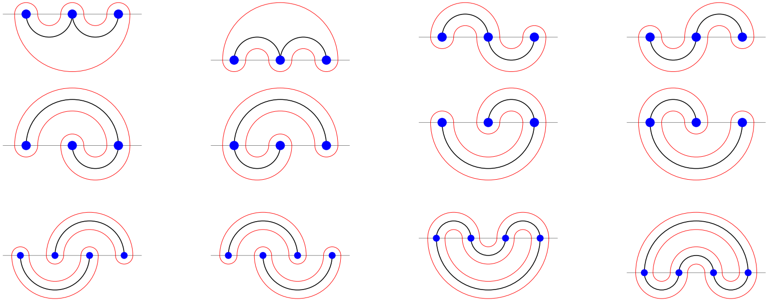

Figure: All meanders in $I_{n,r=2}$. We have $[Y^2]I(X,Y) = 8X^3 + 4X^4$.

The main theorem

Recall that $M_n^{(s)} = \mathrm{Cat}_n^2 \cdot \mathbb{P}(X_n = s)$ is the number of meandric systems on $2n$ points with s components.

Theorem

For any fixed $r \geq 1$ there exists a polynomial $\tilde{P}_r$ of degree at most $3r-3$ such that the generating function of the number of meanders on $2n$ points with $n-r$ components

$$F_r(t) = \sum_{n=r+1}^{\infty} M_n^{(n-r)} t^n = \sum_{n=r+1}^{\infty} \mathbb{P}(X_n = n-r)\, \mathrm{Cat}_n^2\, t^n,$$

with the change of variables $t = w/(1+w)^2$, reads

$$F_r(t) = \frac{w^{r+1}(1+w)}{(1-w)^{2r-1}} \tilde{P}_r(w).$$

Exact results and asymptotics

With the help of a computer, we can enumerate $I_{n,r}$ for $1 \le r \le 6$ (we just have to look at $NC(\le 12)$ to do this) to find

$$\tilde{P}_1(w) = 2$$

$$\tilde{P}_2(w) = 4w^3 - 12w^2 + 4w + 8$$

$$\tilde{P}_3(w) = 18w^6 - 92w^5 + 134w^4 + 8w^3 - 146w^2 + 52w + 42$$

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Corollary

For any fixed $r \geq 1$, assuming that $\tilde{P}_r(1) \neq 0$ (this holds at least for $1 \leq r \leq 6$), the number of meandric systems on $2n$ points having $n - r$ components has the following asymptotic behavior:

$$M_n^{(n-r)} \sim \frac{\tilde{P}_r(1)}{2^{2r-2}\Gamma((2r-1)/2)} 4^n n^{(2r-3)/2}.$$

Equivalently, $\mathbb{P}(X_n = n - r) \sim c_r 4^{-n} n^{(2r+3)/2}$.

Thank you!

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