A permutation model for free random variables

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Voiculescu's definition

Free probability theory

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Non-commutative probability theory

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Free independence

Non-commutative probability spaces

Definition

A non-commutative probability space *(ncps)* is a couple $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional with $\varphi(1) = 1$.

Elements $x \in \mathcal{A}$ are called random variables.

Examples

- $(L^{\infty}(\Omega, \mathbb{P}), \mathbb{E})$;
- $L^{\infty -}(\Omega, \mathbb{P}) = \bigcap_{1 \leqslant p < \infty} L^p(\Omega, \mathbb{P})$;
- $\mathcal{M}_n(\mathbb{C})$ with the normalized trace $\mathrm{tr}_n(A) = \frac{1}{n} \mathrm{Tr}(A)$;
- The group algebra $\mathbb{C}[G]$ with the state

$$\varphi\left(\sum_{g \in G} x_g g\right) = x_e,$$

  where e is the neutral element of G.

Free independence

Definition

Let $(\mathcal{A}, \varphi)$ be a ncps. Unital subalgebras $\mathcal{A}_1, \cdots, \mathcal{A}_n, \ldots$ are called freely independent (or free), if

$$\varphi(a_1 a_2 \cdots a_k) = 0$$

whenever we have

- $a_j \in \mathcal{A}_{i(j)}$ for all $j = 1, \ldots, k$;
- $\varphi(a_j) = 0$ for all $j = 1, \ldots, k$;
- $i(1) \neq i(2)$, $i(2) \neq i(3)$, ..., $i(k-1) \neq i(k)$.

Random variables (x_i) in $\mathcal{A}$ are called free if their generated unital algebras are free.

Free independence

Free independence is a rule for computing mixed moments.

Examples

- If a and b are free random variables, then

$$\varphi\left[(a-\varphi(a)1)(b-\varphi(b)1)\right]=0,$$

  which implies

$$\varphi(ab)=\varphi(a)\varphi(b).$$

- Similarly, if the families $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are free, then

$$\begin{aligned}\varphi(a_1b_1a_2b_2) =& \varphi(a_1a_2)\varphi(b_1)\varphi(b_2)+\varphi(a_1)\varphi(a_2)\varphi(b_1b_2)\\ &-\varphi(a_1)\varphi(a_2)\varphi(b_1)\varphi(b_2).\end{aligned}$$

Convergence in distribution and the free CLT

Definition

Let $(\mathcal{A}_n, \varphi_n)_{n\in N}$ and $(\mathcal{A}, \varphi)$ be ncps and consider random variables $a_n \in \mathcal{A}_n$ and $a \in \mathcal{A}$. We say that a_n converges in distribution towards a, and we write $a_n \xrightarrow{\mathrm{d}} a$, if $\lim_{n\to\infty} \varphi_n(a_n^k) = \varphi(a^k)$ for all $k \geqslant 1$.

Theorem (free Central Limit Theorem)

Let $(a_n)_n$ be a sequence of free, identically distributed random variables such that $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$. Then

$$\frac{a_1 + \cdots + a_n}{\sqrt{n}} \xrightarrow{d} s,$$

where s is a standard semicircular random variable. s has distribution

$$\varphi(s^n) = \int_{-2}^{2} t^n \frac{1}{2\pi} \sqrt{4 - t^2} dt.$$

Gaussian random matrices

Definition

A Gaussian random matrix is a matrix $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{C})$ with random elements such that

- A is self-adjoint: $a_{ij} = \overline{a}_{ji}$ for all i, j;
- $\{a_{ii}\}_{1 \leqslant i \leqslant n}$ are independent real Gaussian random variables with mean 0 and variance $1/n$;
- $\{\Re a_{ij}\}_{1 \leqslant i < j \leqslant n}$ and $\{\Im a_{ij}\}_{1 \leqslant i < j \leqslant n}$ are independent real Gaussian random variables with mean 0 and variance $1/2n$.

Gaussian random matrices can be seen as random variables in the ncps $(\mathcal{M}_n(L^{\infty-}(\Omega, \mathbb{P})), \mathrm{tr} \otimes \mathbb{E})$, where

$$\mathrm{tr} \otimes \mathbb{E}(A) = \mathbb{E}[\mathrm{tr}(A)] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[a_{ii}].$$

Voiculescu's theorem

Theorem

Let (A_n) and (B_n) be independent sequences of Gaussian random matrices. Then

$$(A_n, B_n) \xrightarrow{d} (s_1, s_2),$$

where s_1 and s_2 are free standard semicircular random variables. In particular, we say that independent Gaussian random matrices are asymptotically free.

How about non-random (deterministic) matrices ?
Can one construct deterministic matrices $(A_n, B_n, C_n, \ldots)$ which are asymptotically free ?

P. Biane ('95): Yes, on the group algebra of the symmetric group.

The set-up

Consider the group $\mathcal{S}$ of finitely supported permutations on the set of nonnegative integers $\mathbb{N} = \{0, 1, \ldots\}$.
The ncps we shall work with is the group algebra $\mathbb{C}[\mathcal{S}]$ together with its canonical trace

$$\varphi\left(\sum_{\sigma} x_\sigma \sigma\right) = x_e,$$

where e is the identity permutation.
For all $r, n \geqslant 1$ and $t \in [0, \infty)$, define the random variables

$$M_r(n,t) = \frac{1}{n^{r/2}} \sum \underbrace{(0 a_1 a_2 \cdots a_r)}_{\substack{\text{designs the cycle} \\ 0 \to a_1 \to a_2 \to \cdots \to a_r \to 0}} ,$$

where the sum runs over all r-uplets $(a_1, \ldots, a_r)$ of pairwise distinct integers of $[1, nt]$.

The result

Theorem

The non-commutative distribution of the family $(M_r(n,t))_{r\geqslant 1, t\in[0,+\infty)}$ converges, as n goes to infinity, to the one of a family $(M_r(t))_{r\geqslant 1, t\in[0,+\infty)}$ such that

- $(M_1(t))_{t\in[0,+\infty)}$ *is a free Brownian motion (Biane's result);*
- *for all r, t, one has*

$$M_r(t) = t^{\frac{r}{2}} U_r(t^{-1/2} M_1(t)),$$

where the U_r's are the Chebyshev polynomials of second kind.

The free Brownian motion

Definition

Let $(\mathcal{A}, \varphi)$ be a ncps. A family of random variables $(B_t)_{t \geqslant 0}$ is called a free Brownian motion if

- $B_0 = 0$;
- For all $s \leqslant t$, $B_t - B_s$ is free with the unital algebra generated by $\{B_u, u \leqslant s\}$;
- For all $s \leqslant t$, $B_t - B_s$ has a semicircular distribution with mean 0 and variance $t - s$.

Chebyshev polynomials of second kind

- $U_0(x) = 1, U_1(x) = x, U_2(x) = x^2 - 1, U_3(x) = x^3 - 2x,$ etc.
- U_n is a degree n polynomial defined by

$$U_n(2\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \forall n \geqslant 0.$$

- They satisfy the recurrence relation

$$U_1(x)U_n(x) = U_{n-1}(x) + U_{n+1}(x), \quad \forall n \geqslant 1.$$

- They are orthogonal on $[-2, 2]$ with respect to the semicircular (!) weight

$$w(x) = \frac{1}{2\pi}\sqrt{4 - x^2}.$$

Asymptotically free deterministic matrices

Fix $r = 1$ and consider the elements

$$A(n) = \frac{1}{\sqrt{n}} \sum_{a=1}^{n} (0a) \quad \text{and} \quad B(n) = \frac{1}{\sqrt{n}} \sum_{b=n+1}^{2n} (0b).$$

By the main theorem, the family $(A(n), B(n))$ converges in distribution to a free family $(s_1 = B_1, s_2 = B_2 - B_1)$ of standard semicircular elements. Hence, $A(n)$ and $B(n)$ are asymptotically free.

$A(n)$ and $B(n)$ act by right multiplication on the group algebra $\mathbb{C}[\mathcal{S}]$. The (finite-dimensional) matrices of these operators have the same joint distribution as $(A(n), B(n))$.

Classical vs. Free probability

Free probability has been constructed in a deep analogy with classical probability theory. There is a "dictionary" between the objects of the two theories:

Classical Probability	Free Probability
classical (or tensor) independence	free independence
Gaussian distribution	semicircular distribution
(general) partitions	non-crossing partitions
classical cumulants	free cumulants, *etc.*

Does our model of permutations have a classical-probability analogue ?

The classical model

Recall the definition of the random variables used in the free setting

$$M_r(n,t) = \frac{1}{n^{r/2}} \sum (0 a_1 a_2 \cdots a_r)$$

Idea: replace permutations by sets:

$$L_r(n,t) = \frac{1}{n^{r/2}} \sum \{a_1, a_2, \ldots, a_r\}$$

The classical model

Let $\mathcal{G}$ be the group of finite sets of positive integers endowed with the symmetric difference operation Δ. Consider the commutative ncps $(\mathbb{C}[\mathcal{G}], \psi)$, where ψ is the canonical trace defined by

$$\psi\left(\sum_A x_A A\right) = x_\emptyset.$$

For all $r, n \geqslant 1$ and $t \in [0, \infty)$, define the random variables

$$L_r(n,t) = \frac{1}{n^{r/2}} \sum \{a_1, a_2, \ldots, a_r\},$$

where the sum runs over all r-uplets $(a_1, \ldots, a_r)$ of pairwise distinct integers of $[1, nt]$.

The main theorem in the classical case

Theorem

The (non-commutative) distribution of the family $(L_r(n,t))_{r\geqslant 1, t\in[0,+\infty)}$ converges, as n goes to infinity, to the one of a family $(L_r(t))_{r\geqslant 1, t\in[0,+\infty)}$ such that

- $(L_1(t))_{t\in[0,+\infty)}$ *is a (classical) Brownian motion;*
- *for all r, t, one has*

$$L_r(t) = t^{\frac{r}{2}} H_r(t^{-1/2} L_1(t)),$$

  where the H_r's are the Hermite polynomials.

Remark: The Hermite polynomials are orthogonal with respect to the Gaussian distribution.

Combinatorics and free probability

- $M_{r=1}(t=1)$ is a standard semicircular random variable; $L_{r=1}(t=1)$ is a standard Gaussian. Their moments are given by
$$\varphi(M_1(1)^{2n}) = C_n = \frac{1}{n+1}\binom{2n}{n}$$
and,
$$\psi(L_1(1)^{2n}) = (2n)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

- What about the moments of $M_r(1)$ and $L_r(1)$ for $r \geqslant 2$?

- Combinatorial approach to free probability: free cumulants, non-crossing partitions, etc.

Some insight from classical probability

Let X and Y be two independent classical random variables. What are the moments of $X+Y$?

$$\mathbb{E}\left[(X+Y)^n\right] = \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}[X^k Y^{n-k}] = \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}[X^k]\mathbb{E}[Y^{n-k}].$$

Idea: use Fourier transform !

$$\mathcal{F}_{X+Y}(t) = \mathcal{F}_X(t) \cdot \mathcal{F}_Y(t).$$

Write $\log \mathcal{F}_Z(t) = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} c_n(Z)$. If X and Y are independent, then

$$c_n(X+Y) = c_n(X) + c_n(Y).$$

The quantities c_n are called classical cumulants.

Free cumulants

The free cumulants κ_n are multilinear functionals $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ defined by

$$\varphi(a_1 a_2 \cdots a_n) = \sum_{\sigma \in NC(n)} \kappa_\sigma(a_1, a_2, \ldots, a_n),$$

where

- $NC(n)$ is the lattice of non-crossing partitions of $\{1, \ldots, n\}$. A partition π is non-crossing if there are no $i < j < k < l$ such that $i \stackrel{\pi}{\sim} k$ and $j \stackrel{\pi}{\sim} l$.
- κ_σ is defined as a product over the blocks of σ.

Theorem

If a and b are free random variables, then

$$\kappa_n(a+b, \ldots, a+b) = \kappa_n(a, \ldots, a) + \kappa_n(b, \ldots, b) \quad \forall n \geqslant 1$$

Notation: $NC_2(n)$ is the set of non-crossing pairings of n.

Moments and free cumulants of the family M_r

Fix $t = 1$ and let $M_r = M_r(1)$. Let $\vec{r} = (r_1, \ldots, r_p)$ be a vector of positive integers and put $|\vec{r}| = r_1 + \ldots + r_p$. Consider the following sets of non-crossing pairings:

$$NC_2(\vec{r}) = \{\pi \in NC_2(|\vec{r}|) \mid \pi \wedge \hat{1}_{\vec{r}} = \hat{0}_{|\vec{r}|}\} \text{ and}$$
$$NC_2^*(\vec{r}) = \{\pi \in NC_2(\vec{r}) \mid \pi \vee \hat{1}_{\vec{r}} = \hat{1}_{|\vec{r}|}\}.$$

Theorem

The distribution of the family $(M_r)_{r \geqslant 1}$ is characterized by the fact that its mixed moments are given by

$$\varphi(M_{r_1} M_{r_2} \cdots M_{r_p}) = \# NC_2(\vec{r})$$

and its free cumulants are given by

$$\kappa_p(M_{r_1}, M_{r_2}, \ldots, M_{r_p}) = \# NC_2^*(\vec{r}).$$

Moments and classical cumulants of the family L_r

Fix $t = 1$ and let $L_r = L_r(1)$. Let $\Pi_2(\vec{r})$ be the set of general (i.e. possibly crossing) pairings π of $\{1, \ldots, |\vec{r}|\}$ such that $\pi \wedge \hat{1}_{\vec{r}} = \hat{0}_{|\vec{r}|}$ and $\Pi_2^*(\vec{r}) = \{\pi \in \Pi_2(\vec{r}) | \pi \vee \hat{1}_{\vec{r}} = \hat{1}_{|\vec{r}|}\}$.

Theorem

The distribution of the family $(L_r)_{r \geqslant 1}$ is characterized by the fact that its mixed moments are given by

$$\psi(L_{r_1} L_{r_2} \cdots L_{r_p}) = \#\Pi_2(\vec{r})$$

and its classical cumulants are given by

$$c_p(L_{r_1}, L_{r_2}, \ldots, L_{r_p}) = \#\Pi_2^*(\vec{r}).$$

Conclusion

- We generalized Biane's result beyond transpositions
- Constructed a classical-probability analogue of the model
- More lines added to the "dictionary":

Classical Probability	Free Probability
$\dots$	$\dots$
subsets, Δ	symmetric group
$L_r(t)$	$M_r(t)$
Hermite polynomials	Chebyshev 2^{nd} kind

- Some interesting combinatorics

Perspectives

- Recently, we noticed that $L_r(t)$ is the r-th multiple stochastic integral of a Brownian motion. Michael Anshelevich studied similar questions in the free case *(work in progress)*.
- Are there analogous models for other types of independence in non-commutative probability theory (boolean, monotone) ?
- The moments of the random variables M_r count some particular semi-standard Young tableaux. Is there a connection with representation theory ?

Thank you !

http://arxiv.org/abs/0801.4229