Majorization, entanglement catalysis, stochastic domination and ℓ_p norms

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LOCC transformations & majorization

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Question

Under what conditions can Alice and Bob realize the LOCC transformation

$$\varphi_{AB} \to \psi_{AB} \quad ?$$

Nielsen's result

- Consider Schmidt decompositions for φ (the input state) and ψ (the target state):

$$\varphi = \sum_{i=1}^{d} \sqrt{x_i}\, a_i \otimes b_i,$$

$$\psi = \sum_{i=1}^{d} \sqrt{y_i}\, c_i \otimes d_i.$$

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- Only Schmidt vectors x and y appear in the condition; Alice and Bob can change basis locally.

The majorization relation $\prec$

- Consider $P_d = \{x \in \mathbb{R}^d | x_i \geqslant 0 \text{ and } \sum x_i = 1\}$, the simplex of probability vectors of size d. We have $P_d \subset P_{d+1} \subset \cdots$.

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Definition

Let $x, y \in P_d$. We say that x is majorized by y ($x \prec y$) iff

$$\forall k = 1, 2, \ldots, d \quad \sum_{i=1}^{k} x_i^{\downarrow} \leqslant \sum_{i=1}^{k} y_i^{\downarrow}.$$

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- $(1/d, 1/d, \ldots, 1/d) \rightsquigarrow$ maximal entangled state: anything can be obtained from a maximally entangled input.
- $(1, 0, \ldots, 0) \rightsquigarrow$ separable state: only separable states can be obtained starting with a separable state.

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Proposition

For two probability vectors $x, y \in P_d$, the following assertions are equivalent:

1. $x \prec y$,
2. $\forall t \in \mathbb{R}, \quad \sum_{i=1}^{d} |x_i - t| \leqslant \sum_{i=1}^{d} |y_i - t|$,
3. *There exists a bistochastic matrix B such that $x = By$,*
4. $x \in S_d(y) = \{(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(d)}) \mid \sigma \in \mathcal{S}_d\}$.

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- Converse is false !!! Two manifestations:
  1. Entanglement catalysis
  2. Multiple-copy transformations.

Entanglement catalysis

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Definition

Let $x, y \in P_d$. We say that x is *ELOCC*-majorized (or trumped) by y ($x \prec_T y$) if there exists a probability vector $z \in P_k$ such that

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Definition

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More complicated relations

- For $y \in P_d$, we introduce the following sets:

$$T_d(y) = \{x \in P_d \mid x \prec_T y \Leftrightarrow \exists z \in P_k \text{ s.t. } x \otimes z \prec y \otimes z\},$$
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- For all y, $S_d(y) \subseteq M_d(y) \subseteq T_d(y)$. For the last inclusion, use catalyst

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Question

Provide "nice" descriptions of $\overline{M_d(y)}$ and $\overline{T_d(y)}$. Does $\overline{M_d(y)} = \overline{T_d(y)}$?

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- Are those conditions sufficient ?

Main results

- Let $y \in P_d$, $y_{\min} > 0$.
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Theorem (Aubrun + N. '07)

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Theorem (Turgut '08)

(A)+(B)+(C) $\Leftrightarrow x \in \overline{T_d(y)}$.

The proof

From vectors to atomic measures

- The main idea of the proof (cf. G. Kuperberg): to a probability vector $x \in P_d$, associate a probability measure

$$\mu_x = \sum_{i=1}^{d} x_i \delta_{\log x_i}.$$

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Definition (Stochastic domination)

Let μ and ν be probability measures.

$$\mu \leqslant_{\text{st}} \nu \Leftrightarrow \mu[t,\infty) \leqslant \nu[t,\infty) \quad \forall t \in \mathbb{R}.$$

Equivalently, $\mu \leqslant_{\text{st}} \nu$ iff. there exist some realizations $X \sim \mu$, $Y \sim \nu$ such that $X \leqslant Y$ almost surely.

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- Cons:
  1. The converse of the previous implication does not hold: the two relations are not equivalent.
  2. In particular,

     $$\mu_x \leqslant_{\text{st}} \mu_y \Rightarrow \text{size}(x) > \text{size}(y).$$

ℓ_p norms, Laplace transform and large deviations

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Theorem (Cramér's large deviations theorem)

Let X be a r.v. and assume $\Lambda(\lambda) := \log \mathbb{E}e^{\lambda X} < +\infty$. Introduce Λ^*, the Legendre transform of Λ:

$$\Lambda^*(t) = \sup_{\lambda \in \mathbb{R}} \lambda t - \Lambda(\lambda).$$

Then, for all $t \in (\min X, \max X)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_1 + \cdots X_n \geqslant nt) = \begin{cases} 0 & \text{if } t \leqslant \mathbb{E}X, \\ -\Lambda^*(t) & \text{if } t \geqslant \mathbb{E}X, \end{cases}$$

where $X_1, X_2, \ldots$ denote i.i.d. copies of X.

Stochastic domination for sums of i.i.d. r.v.

Corollary

Consider two random variables X and Y such that

1. $\forall\lambda > 0$, $\mathbb{E}e^{\lambda X} < \mathbb{E}e^{\lambda Y} < \infty$;
2. $\forall\lambda < 0$, $\mathbb{E}e^{\lambda Y} < \mathbb{E}e^{\lambda X} < \infty$;
3. $\mathbb{E}X < \mathbb{E}Y$;
4. $\max X < \max Y$;
5. $\min X < \min Y$.

Then, there exists an integer N such that for all $n \geqslant N$,

$$X_1 + \cdots + X_n \leqslant_{st} Y_1 + \cdots + Y_n,$$

where $X_1, X_2, \ldots$ and resp. $Y_1, Y_2, \ldots$ are i.i.d. copies of X resp. Y.

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- Note that the result fails if we replace strict inequalities by large ones.

Sketch of proof

- Define

$$f_n(t) := \mathbb{P}(X_1 + \cdots + X_n \geqslant nt),$$
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- Cramér's theorem gives limits for $\frac{1}{n}\log f_n$ on $[\mathbb{E}X, \max X]$ and for $\frac{1}{n}\log(1 - f_n)$ on $[\min X, \mathbb{E}X]$. Idem for g_n and Y.

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- Strict Legendre transform inequalities $\Rightarrow$ strict inequalities for the limit functions.
- Since limits are continuous and monotone on a compact set, the inequality $f_n \leqslant g_n$ should hold uniformly for some finite n.

In terms of majorization

Corollary

Consider two probability vectors $x \in P_{d_x}$ *and* $y \in P_{d_y}$ *such that*

1. $\forall\ 1 < p < +\infty,\ N_p(x) < N_p(y)$;
2. $\forall\ -\infty < p < 1,\ N_p(x) > N_p(y)$;
3. $H(x) > H(y)$; *(note that* $\mathbb{E} V_x = -H(x)$*)*
4. $x_{\max} < y_{\max}$;
5. $x_{\min} < y_{\min}$.

Then, there exists an integer N *such that for all* $n \geqslant N$, $x^{\otimes n} \prec y^{\otimes n}$. *In other words,* $x \in M_{d_x}(y)$.

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- From this corollary, one can deduce our two theorems by replacing approximating x by

$$\left(x_1 - \frac{\varepsilon}{d}, \dots, x_d - \frac{\varepsilon}{d}, \frac{\varepsilon}{k}, \dots, \frac{\varepsilon}{k}\right) \text{ or } \left(x_1 - \frac{\varepsilon}{d}, \dots, x_d - \frac{\varepsilon}{d}, \varepsilon\right),$$

for small enough ε (and large enough k).

Thank you !

`http://arxiv.org/abs/quant-ph/0702153`
Comm. Math. Phys. 278 (2008), no. 1, 133-144

and

`http://arxiv.org/abs/0707.0211`
to appear in Ann. Inst. H. Poincaré Probab. Statist.