Tensor norms for quantum information theory

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Problem Statement

Given a set of m Banach spaces $(X_1, \|\cdot\|_1), \ldots, (X_m, \|\cdot\|_m)$, a norm $\|\cdot\|$ on $X = X_1 \otimes \cdots \otimes X_m$ is called a *tensor norm* (or a cross norm) if, for all simple tensors $x = x_1 \otimes \cdots \otimes x_m$, we have $\|x\| = \|x_1\|_1 \cdots \|x_m\|_m$. It has been known since the pioneering work of Grothendieck [Gro56] that there is a smallest and a largest such norms, namely the injective and the projective tensor norms

$$\|x\|_\varepsilon := \sup\{\langle \varphi_1 \otimes \cdots \otimes \varphi_m, x\rangle \,:\, \varphi_i \in X_i^*, \|\varphi_i\|_i^* \leq 1\}$$
$$\|x\|_\pi := \inf\left\{\sum_i \|x_1^{(i)}\|_1 \cdots \|x_m^{(i)}\|_m \,:\, x = \sum_i x_1^{(i)} \otimes \cdots \otimes x_m^{(i)}\right\}.$$

For example, when identifying $\mathbb{C}^d \otimes \mathbb{C}^d$ with the space of $d \times d$ matrices, the injective norm corresponding to the ℓ_2 norms on $\mathbb{C}^d$ corresponds to the operator norm, while the projective norm corresponds to the Schatten 1-norm. For more than two tensor factors, it is known that it is NP-hard to compute these norms [HL13, FL18]. In quantum information theory, most of the time the relevant Banach space is the Schatten 1 space $(\mathcal{M}_d(\mathbb{C}), \|\cdot\|_{tr})$, see [AS17]. It is a folklore result that a quantum state $\rho \in \mathcal{M}_{d_1} \otimes \cdots \otimes \mathcal{M}_{d_m}$ is *separable* if $\|\rho\|_\pi = \operatorname{Tr} \rho = 1$, where the projective norm is relative to the Schatten 1-norms (or trace norms) on the tensor factors.

Goal of the internship

The candidate will study several recent proposals [XQLJ18, MS18] for algorithms approximating the projective (or nuclear) norm, with goal of applying such results to the problem of quantum entanglement [HHHH09]. Working in different scenarios (pure states, symmetric states, low rank states, general mixed states), the different methods for computing the tensor nuclear norm will be analyzed from the point of view of entanglement/separability guarantees, and computational complexity (to be compared with that of traditional methods, such as sum-of-squares hierarchies [DPS04] and ε-nets). The use of different *tensor decompositions* [KB09] in quantum information theory is a different topic which will be explored.

Candidate's profile

The candidate should have a strong mathematical profile. Competences in linear and multilinear algebra, operator algebras, and quantum (information) theory are the most important for the research project (but not strictly required).

References

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[MS18] Andrea Montanari and Nike Sun. Spectral algorithms for tensor completion. *Communications on Pure and Applied Mathematics*, 71(11):2381–2425, 2018.

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