COMPATIBILITY OF QUANTUM MEASUREMENTS

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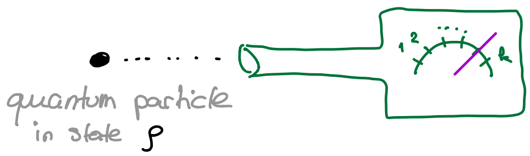
① Compatibility of quantum measurements

→ quantum states ≡ density matrices $\{\rho \in M_d(\mathbb{C}) : \rho \geq 0 \text{ and } \operatorname{Tr}\rho = 1\}$ ← PSD order

→ measurement results in QM are random



measurement apparatus produces an outcome $i \in \{1, 2, \dots, k\}$ with probabilities $(p_1, \dots, p_k)$
$p_i \geq 0$, $\sum p_i = 1$

Axioms of QM: the map $\rho \longmapsto p = (p_1, \dots, p_k)$ must be:

- linear
- positive $p = (p_1, \dots, p_k) \in \mathbb{R}_+^k$
- normalized $\sum p_i = 1$

$\Rightarrow$ $p_i = \operatorname{Tr}(A_i \rho)$ for $A_i \in M_d(\mathbb{C})$ s.t. $A_i \geq 0$ and $\sum A_i = I_d$

Def A k-valued POVM is a k-tuple $A = (A_1, \dots, A_k) \in M_d(\mathbb{C})^k$ s.t. $A_i \geq 0$ and $\sum_{i=1}^{k} A_i = I_d$

Example $k=2$: YES/NO measurements. $A = (E, I-E)$ for some $0 \leq E \leq I$.

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \; ; \; \left(\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) \; ; \; \left(\frac{1}{3}I, \frac{2}{3}I\right)$$

Compatibility Can two POVMs A, B be measured "at the same time" ?

Definition Two POVMs A, B are compatible if $\exists$ a third POVM C s.t. $\begin{cases} \forall i & \sum_j C_{ij} = A_i \\ \forall j & \sum_i C_{ij} = B_j \end{cases}$

→ if POVM A has k outcomes and B has l outcomes, C has $k \cdot l$ outcomes

→ deciding whether A and B are compatible is an SDP with $k \cdot l$ variables, $k+l$ constraints, dim d
↳ semidefinite program

→ Two effects E, F are compatible if $\exists\, X_{11}, X_{12}, X_{21}, X_{22} \geq 0$ s.t.

$$\begin{array}{cc} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} & \begin{array}{l} = E \\ = I - E \end{array} \\ \begin{array}{cc} \overset{\shortparallel}{F} & \overset{\shortparallel}{I-F} \end{array} & \end{array}$$

This can be generalized to g-tuples of POVMs: $A^{(1)}, A^{(2)}, \dots, A^{(g)}$ are compatible if

$$\exists\ B = (B_{i_1 \dots i_g}) \text{ s.t. } \forall n \in [g],\ \forall x \in [k_n],\ A^{(n)}_x = \sum_{i_1,\dots,i_{n-1},i_{n+1},\dots,i_g} B_{i_1 \dots i_{n-1} x\, i_{n+1} \dots i_g}$$

Examples

① $A = (a_1 I, \dots, a_k I)$ and $B = (b_1 I, \dots, b_\ell I)$ are compatible: $C_{ij} := a_i b_j I$

↑ these are called trivial POVMs

② $\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$ and $\left(\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right)$ are not compatible

$$\begin{array}{cc} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} & = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \shortparallel & \\ \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & \end{array} \quad \Rightarrow X_{11} = 0, \text{ same for all the others } ↯$$

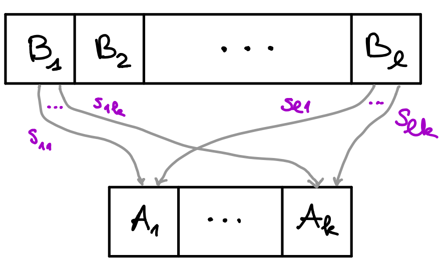
③ If $[A_i, B_j] = 0\ \forall i,j \Rightarrow A$ and B are compatible

take $C_{ij} := A_i B_j = A_i^{1/2} B_j A_i^{1/2} \geq 0$.

② The post-processing relation

Def A POVM $A = (A_1, \dots, A_k)$ is a (classical) post-processing of a POVM $B = (B_1, \dots, B_\ell)$ if $\exists$ (row) stochastic matrix $S \in M_{\ell \times k}$ s.t. $(A_1, \dots, A_k) = (B_1, \dots, B_\ell) \cdot S$

If this is the case, we write $A \prec B$



→ The "$\prec$" relation is a pre-order so we quotient by the relation "$\equiv$":

$$A \equiv B \text{ if } A \prec B \text{ and } B \prec A$$

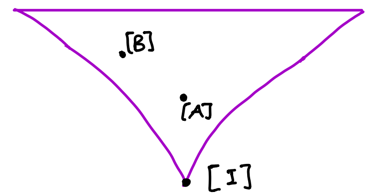
Proposition $A \equiv B$ iff A can be obtained from B by splitting or merging colinear operators

We denote by $[A] = \{ B \text{ POVM} : A \prec B \text{ and } B \prec A \}$. "$\prec$" is an order relation on the equivalence classes.

→ define the length $\ell[A]$ as the minimum # of outcomes of a POVM $B \in [A]$

Example $[I] = \{\text{all trivial POVMs}\} = \{(p_1 I_d, \dots, p_k I_d) : p \text{ probability distribution}\}$

Def let $\mathcal{T} = ([\cdot], \prec)$ the poset of (equivalence classes) of quantum measurements endowed with the post-processing partial order.

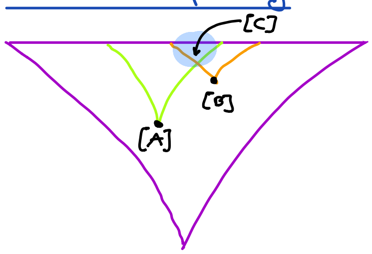


Goal Understand the order-theoretic structure of $\mathcal{T}$

Simple facts

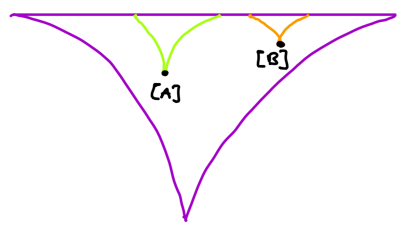
- minimal element : $[I]$ (trivial POVMs) : $\forall A$, $[I] \prec [A]$
- maximal elements : $[A]$ with rk $A_i = 1$ $\forall i$: $[A] \prec [B] \Rightarrow [B] = [A]$

Relation to compatibility : Two POVMs A, B are compatible if $\exists$ POVM C s.t. $A \prec C$ and $B \prec C$



$[A]$ and $[B]$ are compatible

$[C] \succ [A], [B]$



$[A]$ and $[B]$ are not compatible

③ Fisher Information matrix and Zhu's map

- a quantum state ρ can be parametrized as $\rho = I/d + \sum_{i=1}^{d^2-1} \theta_i B_i$, where B_i is a basis of traceless matrices
- For a fixed POVM A, one observes the vector $p_i = \mathrm{Tr}(\rho A_i)$. The corresponding Fisher Information Matrix is [Zhu '15]

$$G_A := \sum_{i=1}^{k} \frac{|A_i \rangle\langle A_i|}{\mathrm{Tr} A_i} \in \mathcal{M}_{d^2}^+ \quad \text{(ignoring the Tr normalization)}$$

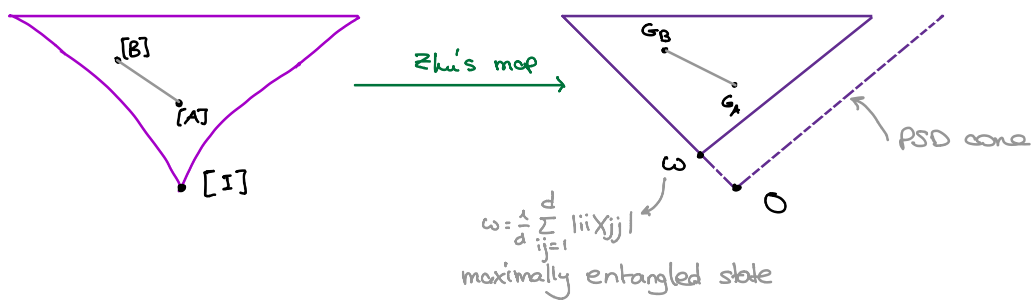
$$\bar{G}_A := \sum_{i=1}^{k} \frac{|\bar{A}_i \rangle\langle \bar{A}_i|}{\mathrm{Tr} A_i} \in \mathcal{M}_{d^2}^+, \quad \text{where } \bar{X} = X - \mathrm{Tr} X \cdot I/d$$

Remark The Zhu map G is a class function $A \equiv B \Rightarrow G_A = G_B$

Proposition [Zhu '15], using [Zamir '98]

The map $G_\cdot : (\mathcal{T}, \prec) \longrightarrow (PSD_{d^2}, \leq)$ is an order morphism : $A \prec B \Rightarrow G_A \leq G_B$

↳ use the fact that $I(\varphi(X); \theta) \leq I(X; \theta)$ ← data processing ineq for Fisher Information



Theorem [Gill-Massar '00, Zhu '15, JHN '19]

For all POVMs A on M_d, $\operatorname{Tr} G_A \leq \min(d, \underbrace{\ell}_{\text{length = min \# of outcomes in the class}}[A])$

Corollary [Zhu '15]

Let $A^{(1)}, \ldots, A^{(g)}$ POVMs on M_d. Define

$$h(A^{(1)}, \ldots, A^{(g)}) := \min \{ \operatorname{Tr} H : H \geq G_{A^{(i)}} \ \forall i = 1, \ldots, g \}$$

If $h > \min\left(d, \prod_{i=1}^{g} \ell[A^{(i)}]\right)$, then $A^{(1)}, \ldots, A^{(g)}$ are incompatible.

→ for 2 POVMs, $h(A,B) = \frac{1}{2}\left[\operatorname{Tr} G_A + \operatorname{Tr} G_B + \| G_A - G_B \|_1\right]$ ← dual SDP for H

→ Why is the incompatibility criterion useful? computational complexity

↳ deciding whether g YES/NO POVMs are compatible → SDP with exp # vars

↳ computing h → SDP with g variables (in dim. d^2)

Remarks

→ $G_\cdot$ is not injective: if A is a 2-design, $G_A = P_{sym}$

→ $G_\cdot$ is faithful on sharp POVMs: $G_A = G_B$ and A, B sharp $\Rightarrow [A] = [B]$

↳ effect operators are projections

Theorem Let $(V, \leq)$ be a partially ordered vector space and $g : PSD_d \to V$ a map s.t.

- $\forall \lambda \geq 0$, $g(\lambda E) = \lambda g(E)$ and
- $g(E+F) \leq g(E) + g(F)$. Then

$$G([A]) := \sum_{i=1}^{\ell[A]} g(A_i)$$ is an order morphism: $A \prec B \Rightarrow G(A) \leq G(B)$

Examples $g(E) = \dfrac{\Phi(E)\Phi(E)^*}{\tau(E)}$ for some map $\Phi : M_d \to M_{r \times c}$ and $\tau : M_d \to \mathbb{C}$ a faithful positive linear form

Proposition Among all quadratic maps, Zhu's map $(\phi(E) = |E\rangle, \tau(E) = \operatorname{Tr} E)$ is the most informative: all other maps can be obtained as $G_A = J G_A^{Zhu} J^*$