On the (in-)compatibility of generic quantum measurements

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Talk outline

Quantum measurements

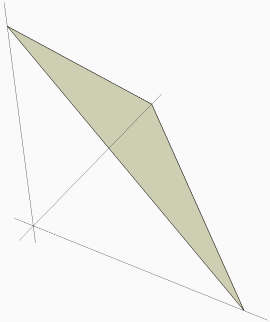
Compatibility

Random POVMs

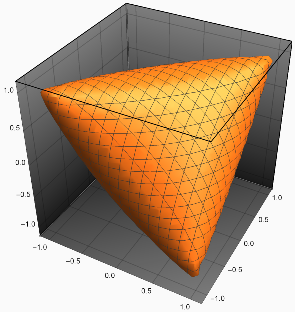
Quantum measurements

Quantum states - the big picture

States	Deterministic	Random mixture
Classical	$x \in \{1, 2, \ldots, k\}$	$\Delta_k = \{p \in \mathbb{R}^k,\ p_i \geq 0,\ \sum_i p_i = 1\}$
Quantum	$\psi \in \mathbb{C}^d,\ \|\psi\| = 1$	$\rho \in \mathcal{M}_d(\mathbb{C}),\ \rho \geq 0,\ \mathrm{Tr}\,\rho = 1$



Classical state space (simplex)



Quantum state space (spectrahedron)

Quantum measurements

- Quantum mechanics postulates: random measurement outcome and collapse of the wave function
- Here, we are only concerned with the measurement outcomes

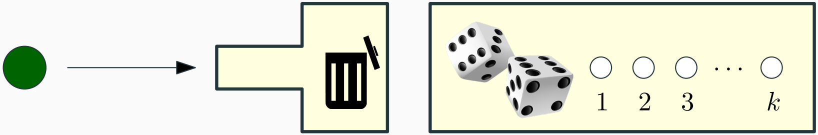
- A quantum measurement is a linear functional $A : \mathcal{D}_d \to \Delta_k$; here d is the dimension (# of degrees of freedom) of the quantum system and k is the number of outcomes of the measurement
- We have $A(\rho) = (\mathrm{Tr}[A_1 \rho], \mathrm{Tr}[A_2 \rho], \ldots, \mathrm{Tr}[A_k \rho])$
- The measurement A is called a POVM (positive-operator-valued measure) and the operators $A_i \in M_d(\mathbb{C})$ are called effects
- They satisfy $A_i \geq 0$ for all $i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} A_i = I_d$

Examples

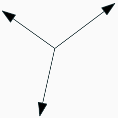
- Measurement in the computational basis: $A_i = E_{ii} = |i\rangle\langle i|$. The probability of obtaining outcome i is $\langle i|\rho|i\rangle = \rho_{ii}$
- Measurement a different basis, e.g.

$$|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$$
$$|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$$

- Trivial measurement: effects are scalar matrices $A_i = p_i I_d$



- General measurement



$$\rightsquigarrow \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 2/3 \end{bmatrix}, \quad A_{2,3} = \begin{bmatrix} 1/2 & \pm\sqrt{3}/6 \\ \pm\sqrt{3}/6 & 1/6 \end{bmatrix}$$

Compatibility

Compatibility

Quantum mechanics predicts that some measurements cannot be performed at the same time

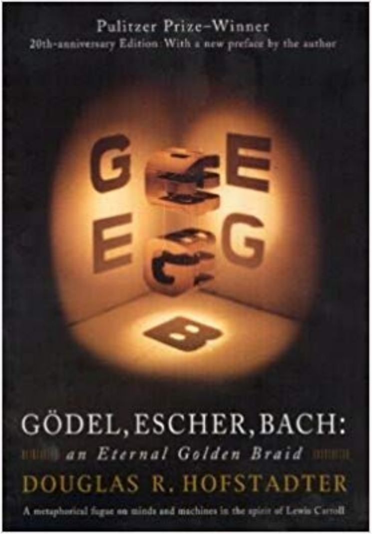
Two quantum measurements

$$A = (A_1, A_2, \ldots, A_k)$$
$$B = (B_1, B_2, \ldots, B_l)$$

are compatible if there exists a third measurement $C = (C_{ij})$ such that

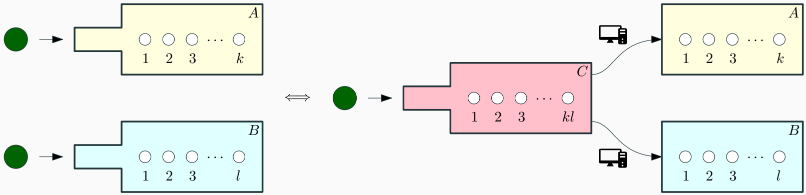
$$\forall i, \quad A_i = \sum_{j=1}^{l} C_{ij}$$

$$\forall j, \quad B_j = \sum_{i=1}^{k} C_{ij}$$



Compatibility

- A, B compatible iff. $\exists C$ POVM s.t. $A_i = \sum_j C_{ij}$ and $B_j = \sum_i C_{ij}$



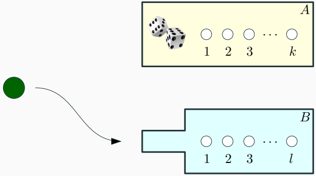
- More generally, POVMs $A^{(1)}, A^{(2)}, \ldots, A^{(g)}$ are compatible iff there exists a POVM $C = (C_{i_1 i_2 \cdots i_g})$ such that

$$\forall n \in [g], \forall x \in [k_n], \quad A_x^{(n)} = \sum_{i_1,\ldots,i_{n-1},i_{n+1},\ldots,i_g} C_{i_1 \cdots i_{n-1}\, x\, i_{n+1} \cdots i_g}$$

Examples

- Measurements in two different bases are incompatible: if $C = (C_{ij})$ is a joint m POVM for $(|e_i\rangle\langle e_i|)_i$ and $(|f_j\rangle\langle f_j|)_j$, then C_{ij} must be a multiple of both $|e_i\rangle\langle e_i|$ and $|f_j\rangle\langle f_j|$
- Trivial measurements are compatible with anything: if $A_i = p_i I_d$,

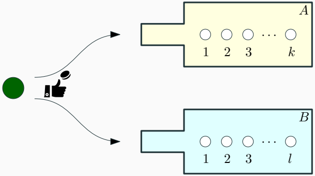
set $C_{i,j} = p_i B_j \geq 0$



- Adding noise produces compatibility: for any POVMs A, B, the noisy versions $\tilde{A}$ and $\tilde{B}$ are compatible

$$\tilde{A} = \left(\frac{1}{2}A_1 + \frac{I_d}{2k}, \ldots, \frac{1}{2}A_k + \frac{I_d}{2k}\right)$$

$$\tilde{B} = \left(\frac{1}{2}B_1 + \frac{I_d}{2l}, \ldots, \frac{1}{2}B_l + \frac{I_d}{2l}\right)$$

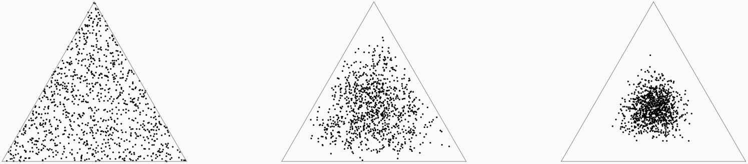


Random POVMs

Random classical and quantum states

- Random classical states: Dirichelt distribution

  $\Delta_d \ni p \sim \mathrm{Dir}(d, n)$ if $\mathbb{P}(p_1, \dots p_d) \sim p_1^{n-1} \cdots p_d^{n-1}$

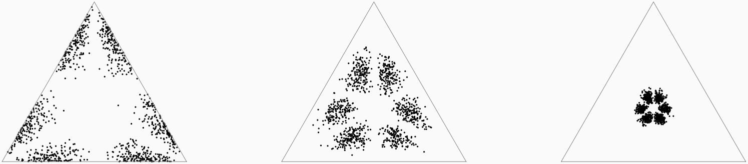


$n = 1$ $n = 3$ $n = 10$

- Random quantum states: normalized Wishart distribution

  $\mathcal{D}_d \ni \rho \sim \mathrm{NormWishart}(d, n)$ if $\rho = W/(\mathrm{Tr}\, W)$ where

  $W = GG^*$, $G \in M_{d \times n}(\mathbb{C})$ with i.i.d. Gaussian entries



$n = 3$ $n = 10$ $n = 100$

- The diagonal of the random quantum states above $\sim \mathrm{Dir}(d, n)$

Random POVMs — definition

- A random POVM of parameters (d, k, n) is a k-tuple of random matrices $(A_1, \ldots, A_k)$ where

$$A_i = S^{-1/2} W_i S^{-1/2}$$

  with $S = \sum_{i=1}^{k} W_i$ and $\{W_i\}_{i=1}^{k}$ is a family of independent Wishart matrices of parameters (d, n)
- Equivalently, we can define

$$A_i = V^*(|i\rangle\langle i| \otimes I_n)V$$

  where $V : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n$ is a Haar-distributed random isometry
- The joint probability distribution reads

$$\mathbb{P}(A_1, \ldots, A_k) \sim \mathbf{1}_{\sum_i A_i = I_d} \prod_{i=1}^{k} \mathbf{1}_{A_i \geq 0} (\det A_i)^{n-d}$$

- If $n = d$, we recover the Lebesgue measure on the convex body of POVMs on $M_d(\mathbb{C})$ with k outcomes

Random POVMs — effect eigenvalues

In the asymptotical regime where k is fixed and $d, n \to \infty$ in such a way that $d \sim tkn$ for some constant $t \in (0, 1]$, the distribution of a random POVM element A_i converges in moments towards the probability measure (Jacobi ensemble)

$$D_t \left[b_{k^{-1}}^{\boxplus t^{-1}} \right] = \max(0, 1 - t^{-1}k^{-1})\delta_0 + \max(0, 1 - t^{-1} + t^{-1}k^{-1})\delta_1 + \frac{\sqrt{(x - \varphi_-)(\varphi_+ - x)}}{2\pi t x (1 - x)} \mathbf{1}_{[\varphi_-, \varphi_+]}(x) \mathrm{d}x,$$

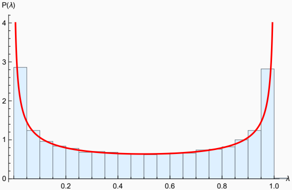
where

$$\varphi_\pm = t + k^{-1} - 2tk^{-1} \pm 2\sqrt{t(1-t)k^{-1}(1-k^{-1})}$$

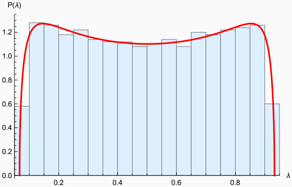
Above, $D.$ is the dilation operator (if X has distribution μ, then aX has distribution $D_a\mu$), b is the Bernoulli distribution ($b_p = (1-p)\delta_0 + p\delta_1$), and $\boxplus$ is the free additive convolution

Effect eigenvalues — simulations vs. theory

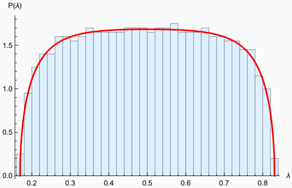
The histogram from each plot corresponds to the eigenvalues of a single sample. Since the first three examples are dichotomic POVMs ($k = 2$), the plots are symmetric with respect to $x = 1/2$



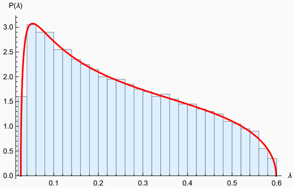
$d = 1000, k = 2, n = 1000$



$d = 1000, k = 2, n = 2000$



$d = 1000, k = 2, n = 4000$



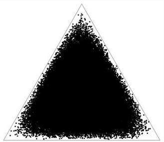
$d = 1000, k = 4, n = 2000$

Random POVMs — probability range

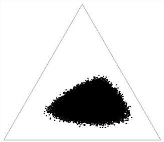
- For any fixed pure quantum state $|\psi\rangle$, the vector $[\langle\psi|M_i|\psi\rangle]_{i=1}^k$ has distribution $\mathrm{Dir}(k, n)$
- ProbRan$(M) := \{[\mathrm{Tr}(\rho M_i)]_{i=1}^k \,:\, \rho \geq 0, \mathrm{Tr}\,\rho = 1\} \in \Delta_k$
- For a sequence of random POVMs $M^{(n)}$ with parameters (d_n, k, n) such that $d_n \sim tkn$ for $t \in (0,1)$, almost surely as $n \to \infty$, ProbRan$(M^{(n)}) \to K_{k,t} \subseteq \Delta_k$, where

$$K_{k,t} := \{\lambda \in \Delta_k \,:\, \forall a \in \Delta_k,\, \langle \lambda, a\rangle \leq \|a\|_{(t)}\}$$

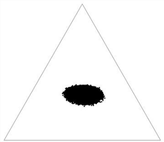
  and the (t)-norm is defined as $\|a\|_{(t)} := \|p_t a p_t\|$, where p_t is a projection of trace t free from a.
- Monte Carlo simulations: $d = k = 3$, 10^5 samples ρ



$n = 1$



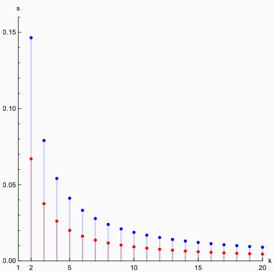
$n = 3$

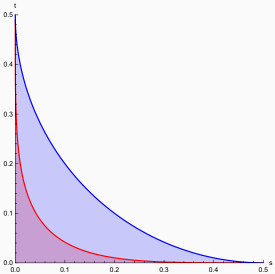


$n = 10$

Random POVMs — compatibility criteria

- Open problem: compute $\lim_{d\to\infty} \mathbb{P}(A \text{ compatible avec } B)$
- Jordan product criterion: if two POVMs A and B are such that for all i, j, $A_i \circ B_j := A_i B_j + B_j A_i \geq 0$, then A and B are compatible
- Noise content criterion: if two POVMs A and B are such that $\sum_i \lambda_{\min}(A_i) + \sum_j \lambda_{\min}(B_j) \geq 1$, then A and B are compatible
- In the asymptotical regime where $d \to \infty$, two independent random POVMs can be certified to be compatible by either the Jordan product or by the noise content criteria





Thank you!

1. T. Heinosaari, M. Jivulescu, I. Nechita - *Random positive operator valued measures* - arXiv:1902.04751, to appear in JMP

2. T. Heinosaari, T. Miyadera, M. Ziman - *An invitation to quantum incompatibility* - Journal of Physics A: Mathematical and Theoretical, 49(12), 123001 (2016)
3. B. Collins, I. Nechita - *Random matrix techniques in quantum information theory* - Journal of Mathematical Physics 57, 015215 (2016)