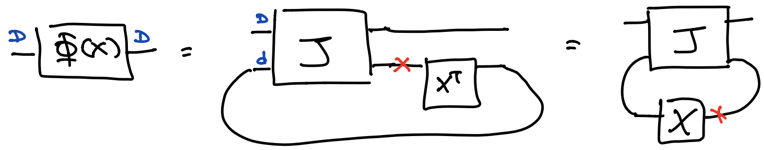
→ Choi Matrix $J(\Phi)$: $M_D \otimes M_d \ni J(\Phi) := [\Phi \otimes id_d](d \cdot \underset{\text{max. entangled state}}{\overset{M_d \otimes M_d}{\omega_d}}) = \sum_{ij=1}^{d} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|$

$\Phi : M_d \longrightarrow M_D$

→ How to recover Φ from $J(\Phi)$: $\Phi(X) = Tr_d [J(\Phi) \cdot I_D \otimes X^T]$ (1)



→ RHS of (1) : $Tr_d [\sum_{ij} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j| \cdot I_D \otimes X^T]$

$= \sum_{ij} \Phi(|i\rangle\langle j|) \cdot Tr(|i\rangle\langle j| \cdot X^T) = \sum_{ij} \Phi(|i\rangle\langle j|) \cdot \langle i|X|j\rangle$

$= \Phi(X)$

Theorem 4.2.1. *Let $\Phi : \mathcal{M}_d \to \mathcal{M}_D$ a linear transformation. The following assertions are equivalent:*

(1) The map Φ is a quantum channel. $\equiv$ TP $\mathrm{Tr}\,\Phi X = \mathrm{Tr}\,X \;\forall X$; CP : $\forall N \geq 1 \; X \geq 0 \Rightarrow (\Phi \otimes \mathrm{id}_N)(X) \geq 0$

(2) The map $\Phi \otimes \mathrm{id}_d$ is positive and trace preserving.
↳ $N = d$ suffices

(3) The Choi map $J(\Phi)$ is positive semidefinite and $\mathrm{Tr}_D\, J(\Phi) = I_d$. ← conditions can easily be checked (on a computer)

(4) The exist R matrices $A_1, \ldots, A_R \in \mathcal{M}_{D\times d}$ such that
Kraus decomposition

$$\Phi(X) = \sum_{i=1}^{R} A_i X A_i^* \qquad \textit{and} \qquad \sum_{i=1}^{R} A_i^* A_i = I_d. \tag{4.14}$$

(5) There exist some positive integer R and an isometry $V : \mathbb{C}^d \to \mathbb{C}^D \otimes \mathbb{C}^R$ such that
Stinespring dilation

$$\Phi(X) = \mathrm{Tr}_R(V X V^*). \tag{4.15}$$

The decomposition (4) above is known as the Kraus decomposition*, while (5) is know as the* Stinespring dilation *of the channel Φ. The integer R above can be taken to be $R = \mathrm{rank}\, J(\Phi)$, value which is called the* Choi rank *of Φ.*

<u>Proof</u> We shall prove $(1) \overset{\text{trivial}}{\Rightarrow} (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$

→ $(2) \Rightarrow (3)$: $J(\Phi) = \underbrace{(\Phi \otimes \mathrm{id}_d)}_{\text{TPCP from (2)}}(d \cdot \omega_d) \geq 0$. $\mathrm{Tr}_D\, (\Phi \otimes \mathrm{id})(d \cdot \omega_d) = d \cdot \underbrace{\mathrm{Tr}_D\, (\Phi \otimes \mathrm{id})(\omega_d)}_{\text{has trace } 1}$

$\Rightarrow$ output has trace d. TP property $\Rightarrow \mathrm{Tr}_D J = I_d$ (see lecture notes)

→ $(3) \Rightarrow (4)$: start from the spectral decomp of J :

$M_D \otimes M_d \ni J = \sum_{i=1}^{R} \lambda_i \, |a_i\rangle\langle a_i|$, where λ_i are the eigenvalues of J $(\lambda_i \geq 0)$
$\mathbb{C}^D \otimes \mathbb{C}^d \ni a_i$ are the eigenvectors

→ $A_i := \sqrt{\lambda_i}\, \mathrm{vec}^{-1}(a_i) \in M_{D\times d}$ $\quad i = 1, 2, \ldots, R$

($\mathbb{C}^D \otimes \mathbb{C}^d \xrightarrow{\mathrm{vec}^{-1}} M_{D\times d}$, $M_{D\times d} \xrightarrow{\mathrm{vec}} \mathbb{C}^D \otimes \mathbb{C}^d$)

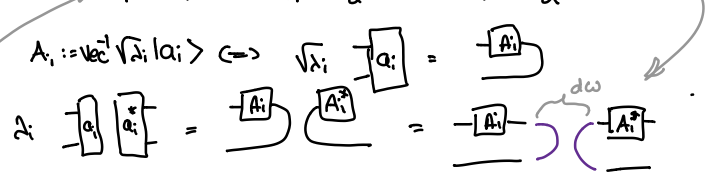
→ $\Phi(X) = \mathrm{Tr}_D\left(J \cdot I_D \otimes X^T\right) = \mathrm{Tr}_D\left(\underbrace{\sum_{i=1}^{R} \lambda_i |a_i\rangle\langle a_i|}_{J} \cdot I_D \otimes X^T\right)$

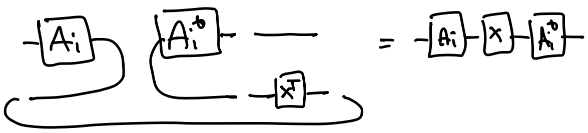
$\overset{(2)}{=} \sum_{i=1}^{R} \mathrm{Tr}_D\left(A_i \otimes I_d \cdot d\omega_d \cdot A_i^* \otimes I_d \cdot I_D \otimes X^T\right)$

$\overset{(3)}{=} \sum_{i=1}^{R} A_i X A_i^*$

(2) at fixed i : $\lambda_i |a_i\rangle\langle a_i| \overset{!}{=} A_i \otimes I_d \cdot d\omega \cdot A_i^* \otimes I_d$

$A_i := \mathrm{vec}^{-1}\sqrt{\lambda_i}|a_i\rangle \iff$



(3) at fixed i : LHS of (3) 

→ for $\sum A_i^* A_i = I_d$ → see lecture notes

→ (4) ⇒ (5) : we have $A_1, \dots, A_R \in M_{D\times d}(\mathbb{C})$. Define $M_{DR\times d} \ni V := \sum_{i=1}^{R} A_i \otimes |i\rangle$

$$V = \begin{bmatrix} \overbrace{A_1}^{d} \\ \vdots \\ A_R \end{bmatrix} \begin{matrix} \}D \\ \\ \end{matrix} \Bigg\} D\cdot R$$

→ $\mathrm{Tr}_R(VXV^*) = \mathrm{Tr}_R\left(\sum_{ij=1}^{R} A_i \otimes |i\rangle \cdot X \cdot \langle j| \otimes A_j^*\right) = \sum_{ij} A_i X A_j^* \cdot \underbrace{\mathrm{Tr}(|i\rangle\langle j|)}_{\delta_{ij}}$

$$= \sum_{i=1}^{R} A_i X A_i^* = \Phi(X)$$

→ $\sum A_i^* A_i = I_d \Longrightarrow V$ is an isometry $(V^*V = I_d)$ ⇝ see lecture notes

→ (5) ⇒ (4) : we want to show $\Phi \otimes id_N$ is TP and Positive

→ let $X \in M_d \otimes M_N$, $X \geq 0$: $(\Phi \otimes id_N)(X) \underset{\text{use (5)}}{\overset{\uparrow}{=}} \mathrm{Tr}_R\left(\underbrace{(V\otimes I_N)X(V\otimes I_N)^*}_{\geq 0}\right) \geq 0$

END of THE PROOF

Back to entanglement

→ the transposition map is positive but NOT completely positive

→ recall that : $(\mathrm{transp}_2 \otimes id_2)(\omega_2) \not\geq 0$

⇒ the transposition is not CP.

Observation

① if $\Phi : M_d \to M_D$ is CP and $\rho \in M_d \otimes M_{d'}$ is positive (e.g. a quantum state)

$$\Rightarrow \underbrace{(\Phi \otimes id_{d'})}_{\text{positive}} \underset{\overset{\vee\!|}{0}}{(\rho)} \geq 0$$

② if $\Psi : M_d \to M_D$ is positive and $\rho \in M_d \otimes M_{d'}$ is separable

$$\Rightarrow \underbrace{(\Psi \otimes id_{d'})}_{\text{can be non-positive in general !}}(\rho) \geq 0$$

→ show ②: ρ separable ⇒ $\rho = \sum_{i=1}^{R} \alpha_i \otimes \beta_i$ with $\alpha_i, \beta_i \geq 0$

$$\Rightarrow (\varphi \otimes \mathrm{id})(\rho) = \sum (\varphi \otimes \mathrm{id})(\alpha_i \otimes \beta_i) = \sum_i \underbrace{\varphi(\alpha_i)}_{\geq 0} \otimes \underbrace{\beta_i}_{\geq 0} \geq 0$$

→ From ② above ⇒ ENTANGLEMENT CRITERION

Proposition If $\varphi : M_d \to M_D$ is a positive map and $\rho \in M_d \otimes M_{d'}$ is a quantum state such that $(\varphi \otimes \mathrm{id}_{d'})(\rho) \ngeq 0$, then ρ is entangled.

$(\varphi \otimes \mathrm{id})(\rho)$ has negative eigs.

→ for $\mathrm{transp} : M_d \to M_d$, $\mathrm{transp}(X) = X^T$, we obtain the positive partial transpose (PPT) criterion : if $\rho^\Gamma := (\mathrm{transp} \otimes \mathrm{id})(\rho) \ngeq 0 \Rightarrow \rho$ entangled

Γ is "half of T"

→ example $\omega_2^\Gamma = \left[\frac{1}{2} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right) \right]^\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{0} & 1 & 0 \\ 0 & 1 & \boxed{0} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ has negative eigs ⇒ ω_2 is ent.

→ **Remark** : The reverse implication is wrong in general : $\begin{cases} \text{if } \rho^\Gamma \geq 0 \Rightarrow \text{cannot conclude} \\ \text{if } \rho^\Gamma \ngeq 0 \Rightarrow \rho \text{ entangled} \end{cases}$

Theorem (Woronowicz)

if $\rho \in M_d \otimes M_{d'}$ with $(d, d') \in \{(2,2), (2,3), (3,2)\}$ is a quantum state, then

$$\rho \text{ entangled} \iff \rho^\Gamma \ngeq 0 \iff (\mathrm{transp} \otimes \mathrm{id})(\rho) \geq 0$$

→ entanglement is characterized by the partial transpose in dim ≤ 6

Theorem let $\rho \in M_d \otimes M_{d'}^{1,+}$. Then ρ separable $\iff (\varphi \otimes \mathrm{id}_{d'})(\rho) \geq 0$ for all positive maps $\varphi : M_d \to M_{d'}$

→ we lack a theory for positive (but not CP) maps. We know that it cannot be easy since detecting separability vs entanglement is NP-hard

Theorem (Woronowicz) $\varphi : M_2 \to M_2$ or M_3 is positive if and only if $\exists\, \Phi_1, \Phi_2$ CP maps such that $\varphi = \Phi_1 + \mathrm{transp} \circ \Phi_2$

→ in other words, the transposition is the only relevant positive but not CP map in small (≤ 6) dimensions