→ physical maps ≡ quantum channels : $\underline{CP}$ & TP

→ positive maps (not necessarily CP) are used to detect entanglement

- if $\rho \in M^{1,+}_{d \cdot d'}$ and $\Psi : M_d \to M_D$ positive is such that $[\Psi \otimes id_{d'}](\rho) \not\geq 0$ then ρ is entangled

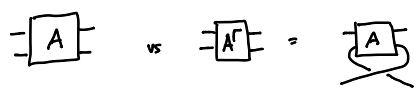
Theorem: If $[\Psi \otimes id_{d'}](\rho) \geq 0$ for all $\Psi : M_d \to M_{d'}$ positive, then ρ is separable.

THE PARTIAL TRANSPOSITION CRITERION (PPT)

→ we focus on the map $\mathrm{transp}_d : M_d \to M_d$, $X \mapsto X^T$

→ transp is positive : $X \geq 0 \Rightarrow X^T \geq 0$ since it has the same eigenvalues

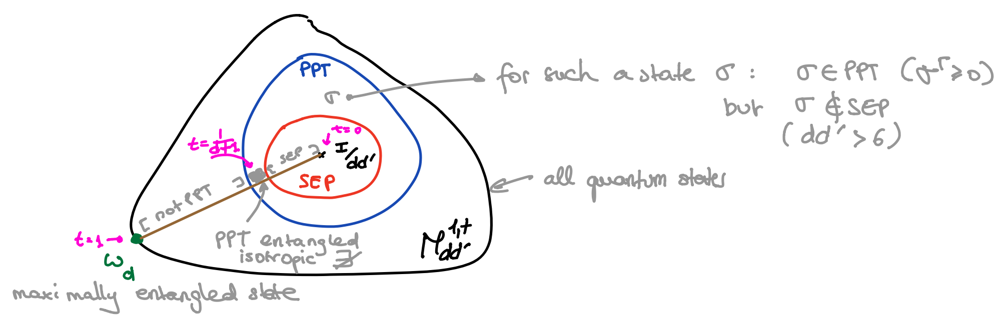
→ transp is not CP $[\mathrm{transp}_2 \otimes id_2](\omega_2) \not\geq 0$

→ We write $A^\Gamma := [id \otimes \mathrm{transp}](A)$ 

→ Define $PPT(d:d') := \{\rho \in M^{1,+}_{dd'} : \rho^\Gamma \geq 0\} = \{$states which "pass" the PPT test$\}$

→ if $\rho \notin PPT \Rightarrow \rho$ entangled . In terms of sets: $SEP(d:d') \subseteq PPT(d:d')$

i.e. ρ did not pass the PPT test



Theorem [Woronowicz] If $(d,d') = (2,2), (2,3), (3,2)$ then $SEP = PPT$!

→ theorem follows from: $\Psi : M_2 \to M_{2,3}$ is positive $\Leftrightarrow$ $\Psi = \Phi_1 + \Phi_2 \circ \mathrm{transp}$, where Φ_1 and Φ_2 are completely positive

→ let $\rho \in M^{1,+}_{dd'}$ be a state s.t. $\rho^\Gamma \geq 0$ ($\Rightarrow \rho \in PPT$) $\overset{!}{\Longrightarrow} \rho \in SEP$

Assume ρ is entangled $\Rightarrow \exists\, \Psi : M_d \to M_{d'}$ positive which detects ρ : $(\Psi \otimes id)(\rho) \not\geq 0$

$$0 \not\leq (\Psi \otimes id)(\rho) = \underbrace{(\Phi_1 \otimes id)(\rho)}_{\geq 0 \text{ b/c } \Phi_1 \text{ is CP}} + \underbrace{(\Phi_2 \circ \mathrm{transp} \otimes id)(\rho)}_{= \underbrace{(\Phi_2 \otimes id)}_{\text{positive map}} \big(\underbrace{\mathrm{transp} \otimes id(\rho)}_{\rho^\Gamma \geq 0} \big)} \geq 0 \quad \text{false} \ \lightning$$

Proposition If $\rho = |\psi\rangle\langle\psi|$ is a pure quantum state, then

$$\rho \text{ is separable} \iff \rho^\Gamma \geq 0 \quad (\rho \text{ is PPT})$$

Proof $\rho = |\psi\rangle\langle\psi|$ with $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d$ a unit vector. Schmidt decomposition for ψ:

$$\psi = \sum_{i=1}^{R} \sqrt{\lambda_i}\; a_i \otimes b_i$$

, where R is the Schmidt rank of ψ,

$\lambda_1, \dots, \lambda_R$ are the Schmidt coeffs $\lambda_i \geq 0$ and $\sum \lambda_i = 1$

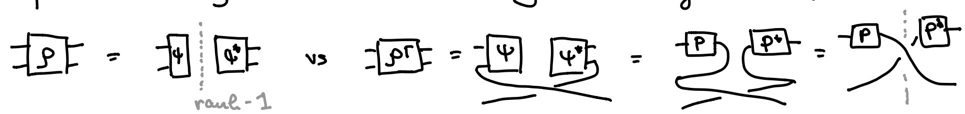
$\{a_i\}_{i=1}^R$ and $\{b_i\}_{i=1}^R$ are $\perp_n$ families of vectors in $\mathbb{C}^d$

→ we have $\rho^\Gamma = |\psi\rangle\langle\psi|^\Gamma = \Big[\sum_{ij=1}^{R} \sqrt{\lambda_i\lambda_j}\; \underbrace{|a_i\rangle|b_i\rangle\langle a_j|\langle b_j|}_{|a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|} \Big]^\Gamma$

$$= \sum_{ij=1}^{R} \sqrt{\lambda_i\lambda_j}\; |a_i\rangle\langle a_j| \otimes \underbrace{\mathrm{transp}(|b_i\rangle\langle b_j|)}_{|\bar{b}_j\rangle\langle \bar{b}_i|}$$

$$= \sum_{i=1}^{R} \sqrt{\lambda_i\lambda_j}\; |a_i\rangle\langle a_j| \otimes |\bar{b}_j\rangle\langle \bar{b}_i|$$

→ ρ was pure (rank 1); ρ^Γ is not rank-1 anymore (in general)!



where $P = \mathrm{vec}^{-1}\ \psi$

→ one can check that ρ^Γ has R^2 non-zero eigenvalues as follows:

- $1 \leq s = t \leq R$ eig λ_s with $\vec{ev}$ $a_s \otimes \bar{b}_s$
- $1 \leq s < t \leq R$ eig $\sqrt{\lambda_s\lambda_t}$ with $\vec{ev}$ $\frac{1}{\sqrt{2}}(a_s \otimes \bar{b}_t + a_t \otimes \bar{b}_s)$
- $1 \leq t < s \leq R$ eig $-\sqrt{\lambda_s\lambda_t}$ with $\vec{ev}$ $\frac{1}{\sqrt{2}}(a_s \otimes \bar{b}_t - a_t \otimes \bar{b}_s)$

$$\rho^\Gamma \cdot a_s \otimes \bar{b}_t = \sum_{ij=1}^{R} \sqrt{\lambda_i\lambda_j}\; |a_i\rangle\underbrace{\langle a_j | a_s\rangle}_{\delta_{js}} \otimes |\bar{b}_j\rangle\underbrace{\langle \bar{b}_i | \bar{b}_t\rangle}_{\delta_{it}}$$

$$= \sqrt{\lambda_t\lambda_s}\; a_t \otimes \bar{b}_s$$

$$\rho^\Gamma \frac{1}{\sqrt{2}}(a_s \otimes \bar{b}_t - a_t \otimes \bar{b}_s) = \sqrt{\lambda_s\lambda_t}\; \frac{1}{\sqrt{2}}(a_t \otimes \bar{b}_s - a_s \otimes \bar{b}_t)$$

$$= -\sqrt{\lambda_s\lambda_t}\; \frac{1}{\sqrt{2}}(a_s \otimes \bar{b}_t - a_t \otimes \bar{b}_s)$$

→ ρ^Γ has negative eigenvalues $\iff R > 1 \iff \psi$ entangled ∎

Isotropic states Convex combinations (mixtures) of ω_d and I/d^2 $(d'=d)$
max. ent max mixed

$$\rho_t = t\cdot\omega_d + (1-t)\frac{I}{d^2} \qquad , t\in[0,1]$$

Proposition PPT criterion is exact for isotropic states

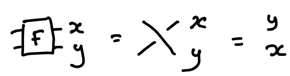
$$\rho_t \in SEP \underset{(1)}{\overset{}{\Longleftrightarrow}} \rho_t \in PPT \underset{(2)}{\Longleftrightarrow} t\in\left[0,\frac{1}{d+1}\right] \quad (3)$$

Proof (1) $\rho_t\in SEP \Rightarrow \rho_t\in PPT$ b/c $SEP\subseteq PPT$

(2) let $\rho_t\in PPT \Leftrightarrow \rho_t^\Gamma \geq 0$. Let us compute $\rho_t^\Gamma = \left[t\omega_d + (1-t)\frac{I}{d^2}\right]^\Gamma$

$$= \frac{t}{d}(d\omega_d)^\Gamma + \frac{1-t}{d^2}\cdot I^\Gamma$$

→ we have $I_{d^2}^\Gamma = (I_d\otimes I_d)^\Gamma = I_d\otimes I_d^T = I_{d^2}$

→  where 

$F(x\otimes y) = y\otimes x$

Flip operator

→ $\rho_t^\Gamma = \frac{t}{d}\cdot F + \frac{1-t}{d^2} I$. What are the eigenvalues of this?

→ eigenvalues of $F_d = \begin{cases} +1 & \text{with mult. dim sym. subspace}(\mathbb{C}^d\otimes\mathbb{C}^d) = \frac{d(d+1)}{2} \\ -1 & \text{with mult. dim anti-sym. subspace } \frac{d(d-1)}{2}\end{cases}$

→ spectrum of $\rho^\Gamma = \left\{\pm\frac{t}{d} + \frac{1-t}{d^2}\right\} \subseteq \mathbb{R}_+ \Leftrightarrow t\leq\frac{1}{d+1}$

(3) If $t\in\left[0,\frac{1}{d+1}\right] \Rightarrow \rho_t\in SEP$. We only need to show it for $t=\frac{1}{d+1}$. Smaller values follow from convexity (of SEP): $s<t$ $\quad \rho_s = s\omega_d + (1-s)\frac{I}{d^2} = \frac{s}{t}\rho_t + \left(1-\frac{s}{t}\right) I/d^2$
quantum states

→ let us fix $t=\frac{1}{d+1}$. We have $\rho_{\frac{1}{d+1}} \overset{(*)}{=} \int_{\substack{x\in\mathbb{C}^d \\ \|x\|=1}} |x\rangle\langle x|\otimes|\bar{x}\rangle\langle\bar{x}| \, d\sigma(x)$,

where $d\sigma$ is the uniform measure on the unit sphere of $\mathbb{C}^d$

→ (*) is an ∞-sum. One can have a finite sum by discretizing the unit sphere.
"spherical designs"
(we need a 2-design)

REDUCTION CRITERION → use $R(X) = (\mathrm{Tr}X)I_d - X$ reduction map

The following remarkable result is due to Gurvits and Barnum; it is a striking result about the geometry of the set of (separable) quantum states. For the proof, see, e.g., [AS17, Theorem 9.15 or Exercise 9.8].

Theorem 3.2.4 ([GB02]). *The largest euclidean ball, centered at $I/(d_1 d_2)$ and contained in $\mathcal{M}^{1,+}_{d_1 d_2}$ is separable.*

→ before giving the proof, we recall some basic facts about norms of operators.

→ for a matrix $A \in \mathcal{M}_{d\times d'}(\mathbb{C})$ having singular values $s_1, \dots, s_r$ we define

- the operator norm : $\|A\| = \|A\|_\infty = \|A\|_{op} := \max_{i=1}^{r} s_i = \sup_{\|x\|=\|y\|=1} |\langle x, Ay\rangle|$
- the Schatten 2-norm (or the Hilbert-Schmidt norm) :

$$\|A\|_2 = \|A\|_{HS} = \left(\sum_{i=1}^{r} s_i^2\right)^{1/2} = \left[\mathrm{Tr}(A^*A)\right]^{1/2}$$

- the Schatten 1-norm (or the trace norm) : $\|A\|_1 = \|A\|_{tr} = \sum_{i=1}^{r} s_i = \mathrm{Tr}\sqrt{A^*A}$

→ we have $\|A\|_{op} \le \|A\|_2 \le \|A\|_1$

→ for the proof, we need some technical lemmas

Lemma 1 For any matrix $M_{d_1} \otimes M_{d_2} \ni A = \sum_{ij=1}^{d_2} A_{ij} \otimes |i\rangle\langle j|$, we have

$$\|A\|^2 \le \sum_{ij=1}^{d_2} \|A_{ij}\|^2$$

Proof Define the block-rows of A by $R_i := \sum_{j=1}^{d_2} A_{ij} \otimes |i\rangle\langle j|$. We have

$$\|R_i R_i^*\| = \left\|\sum_j A_{ij} A_{ij}^* \otimes |j\rangle\langle j|\right\| \le \sum_j \|A_{ij} A_{ij}^*\| = \sum_j \|A_{ij}\|^2 .$$

→ on the other hand, $A = \sum_i R_i$ and $A^*A = \sum R_i^* R_i$ since for $i \neq i'$ $R_i^* R_{i'} = 0$

→ thus, $\|A\|^2 = \|A^*A\| \le \sum_i \|R_i^* R_i\| = \sum_i \|R_i R_i^*\| \le \sum_{ij} \|A_{ij}\|^2$ ∎

Lemma 2 If $\Delta \in M_{d_1 d_2}$ is such that $\|\Delta\|_2 \le 1$, then $I_{d_1 d_2} + \Delta \in \underbrace{\mathbb{R}_+ SEP}_{\text{cone}}$.

cone of separable matrices $A = t\rho$ with $\rho \in SEP$ and $t \ge 0$

Proof We shall use the characterisation of separability by testing with positive maps. We need to show that for any positive and unital map $\varphi : M_{d_1} \to M_{d_2}$, $[\varphi \otimes id](I + \Delta) \ge 0$.

→ we have : $[\varphi \otimes id](\Delta) = I_{d_2^2} + [\varphi \otimes id](\Delta)$, so to conclude it suffices to show

$$\|(\varphi \otimes id)(\Delta)\| = \left\|\sum_{ij} \varphi(\Delta_{ij}) \otimes |i\rangle\langle j|\right\| \le 1$$

→ but $\| \sum_{ij} \Psi(\Delta_{ij}) \otimes |i\rangle\langle j| \|^2 \underset{\text{Lemma 1}}{\leq} \sum_{ij} \| \Psi(\Delta_{ij}) \|^2 \leq \sum_{ij} \| \Delta_{ij} \|^2 \leq \sum_{ij} \| \Delta_{ij} \|_2^2 = \| \Delta \|_2^2 \leq 1$ ∎

positive maps are contractions wrt op. norm

$\| \ \|_{op} \leq \| \ \|_2$

Proof of the theorem

→ we can generalize lemma 2 to : if $M_D \ni A \geq 0$ and $D - \frac{\|A\|_1^2}{\|A\|_2^2} \leq 1$, then $A \in \mathbb{R}_+ SEP(d_1 : d_2)$

↳ indeed, writing $\Delta := sA - I$, the argmin of $s \longmapsto \| sA - I_D \|_2^2$ is $s_0 := \frac{\|A\|_1}{\|A\|_2^2}$

↳ then, $\| s_0 A - I \|_2^2 = D - \frac{\|A\|_1^2}{\|A\|_2^2}$

→ for a quantum state ρ, we have $\| \rho - \frac{I}{D} \|_2^2 = \| \rho \|_2^2 - \frac{1}{D}$ so

$$D - \frac{\overbrace{\|\rho\|_1}^{=1}{}^2}{\|\rho\|_2^2} \leq 1 \iff \| \rho - \frac{I}{D} \|_2^2 \leq \frac{1}{D-1} - \frac{1}{D} = \frac{1}{D(D-1)}$$ ∎