→ random pure states $\quad \int_{\|x\|=1} |x\rangle\langle x| \otimes |\bar{x}\rangle\langle \bar{x}| \; d\sigma(x)$

unif. measure on the unit sphere of $\mathbb{C}^d$

→ random mixed states

PURE STATES $\quad \mathcal{H} = \mathbb{C}^d \qquad x \in \mathbb{C}^d \qquad \|x\| = 1$

→ $x \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ bipartite pure state : $x = \sum_{i=1}^{R} \sqrt{\lambda_i}\; a_i \otimes b_i$ with

"Schmidt decomposition"

- $0 \le \lambda_1, \ldots, \lambda_R \qquad \sum \lambda_i = 1$
- $\{a_i\}_{i=1}^{R}$ orthonormal family in $\mathbb{C}^{d_1}$
- $\{b_i\}_{i=1}^{R}$ ——— " ——— in $\mathbb{C}^{d_2}$

Schmidt rank

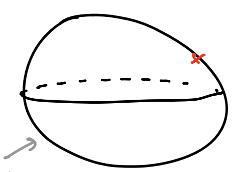
→ $x$ is separable $\iff R = 1$ i.e $x = a \otimes b$

→ if $R \geq 2$, $x$ is entangled

compute Schmidt rank to decide entanglement vs sep.

**Definition** A random pure quantum state is a random uniform point on the unit sphere of $\mathbb{C}^d$

→ $d = 2$




→ Question : if $|\psi\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ is a random bipartite pure q. state,

what is $\mathbb{P}[\, |\psi\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ is separable $]$ ?

$= \mathbb{P}[\, \Psi := \mathrm{vec}^{-1}(|\psi\rangle)$ has rank 1 $]$

$\Psi \in M_{d_1 \times d_2}(\mathbb{C})$

$R$ = schmidt rank of $|\psi\rangle$
$\parallel$
"usual" rank of $\Psi = \mathrm{vec}^{-1}(|\psi\rangle)$

→ $\Psi \in M_{d_1 \times d_2}(\mathbb{C})$ has unit rank $\iff \Psi = |a\rangle\langle b| \iff \forall \, \substack{i<j \\ k<l} \; \det(\Psi_{[i,j|k,l]}) = 0$

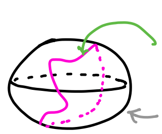
$\iff$ all $2 \times 2$ minors have $\det = 0$

$$\Psi = \begin{bmatrix} \cdot & - & \cdot \\ | & & | \\ \cdot & - & \cdot \end{bmatrix} \begin{matrix} \leftarrow i \\ \\ \leftarrow j \end{matrix}$$
$$\quad \uparrow_k \;\; \uparrow_l$$

$\Psi_{ik}\Psi_{jl} - \Psi_{il}\Psi_{jk} = 0$

→ $\Psi$ has rank 1 $\iff \forall \, \substack{i<j \\ k<l} \; \det(\Psi_{[i,j|k,l]}) = 0$

$\iff$ a finite number of polynomials in the entries of $\Psi$ are 0.

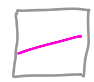
→ 

→ $\{ \psi : \text{rank } \psi = 1 \}$ = sub-manifold of smaller dimension inside the unit sphere of $\mathbb{C}^{d_1 d_2}$

→ $SEP = \{ a \otimes b : a \in \mathbb{C}^{d_1} \ \|a\|=1 \ ; \ b \in \mathbb{C}^{d_2} \ \|b\|=1 \}$ is (path-) connected:

$$a_1 \otimes b_1 \longrightarrow a_2 \otimes b_1 \longrightarrow a_2 \otimes b_2$$

any 2 points on the sphere of $\mathbb{C}^{d_1}$ are connected

→ since SEP has smaller dimension $\Rightarrow$ $\mathbb{P}(|\psi\rangle$ is separable$) = 0$  $\mathbb{P}$(random point in a square being on a line) $= 0$

→ $\mathbb{P}(|\psi\rangle$ entangled$) = 1$

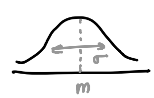
→ we are measuring prob w/ the uniform measure on the unit sphere of $\mathcal{H} = \mathbb{C}^d$

→ in real life: we "see" states with "low energy": ground states of Hamiltonians

→ Hamiltonians we see in real life are not "uniform": they are local → their ground states are not uniform on (the unit sphere of) $\mathcal{H}$.

→ uniform measure on the sphere is not physically meaningfull.

## Gaussian Integration

→ real Gaussian r.v. $\dfrac{dg}{dx} = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$ 

$g \in \mathbb{R}$

→ Gaussian vectors: $m \in \mathbb{R}^d$, $\Sigma$ = covariance matrix $\in M_d^+$

has density $\dfrac{dg}{dx} = \dfrac{1}{(2\pi)^d \det \Sigma} \exp\left[ -\frac{1}{2} \langle x - m, \Sigma^{-1} \cdot (x-m) \rangle \right]$ $\quad g \in \mathbb{R}^d$

→ complex Gaussian : $Z \in \mathbb{C}^d$ has density $\dfrac{1}{\pi^d} \exp\left[ - \dfrac{\|Z\|^2}{2} \right]$ for covariance $= I$
"standard"

→ $\mathbb{E}\, Z = 0 \qquad \mathbb{E}\, Z_i^2 = \mathbb{E}\, \bar{Z}_i^2 = 0 \qquad \mathbb{E}\, Z_i \cdot \bar{Z}_i = \mathbb{E}\, |Z_i|^2 = 1$

→ Important property: invariance under unitary rotation: $Z \in \mathbb{C}^d$ is random, std. Gaussian then, $\forall\, U \in \mathcal{U}_d$, $UZ$ is again std. Gaussian

**Proposition** If $g \in \mathbb{C}^d$ is a random std. Gaussian, $|\psi\rangle := \dfrac{g}{\|g\|}$ is a random uniform point on the unit sphere of $\mathbb{C}^d$, i.e. a random pure quantum state on $\mathbb{C}^d$

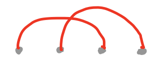
→ this is how we sample in practice a uniform point on the sphere

**Remark** If $g$ is std Gaussian, then $|\psi\rangle := \dfrac{g}{\|g\|}$ and $\|g\|$ are independent random vars.

→ How to compute Gaussian integrals: Wick formula + graphical formalism (boxes + wires)

**Theorem [Wick formula]** If $Z \in \mathbb{C}^d$ is a centered ($m=0$) Gaussian vector, then

$$\forall \underset{1 \le}{i_1}, \dots, \underset{\le d}{i_k} \quad \mathbb{E}[Z_{i_1} Z_{i_2} \cdots Z_{i_k}] = \sum_{\pi \in \underbrace{P_2(k)}_{\text{pairings of } \{1,2,\dots,k\}}} \prod_{(s,t) \in \pi} \mathbb{E}[Z_{i_s} Z_{i_t}]$$

$\pi =$ ($k=4$) 

$$\mathbb{E}[Z_1 Z_2 Z_3 Z_4] = \mathbb{E}[Z_1 Z_2]\,\mathbb{E}[Z_3 Z_4] + \mathbb{E}[Z_1 Z_3]\,\mathbb{E}[Z_2 Z_4] + \mathbb{E}[Z_1 Z_4]\,\mathbb{E}[Z_2 Z_3]$$

→ $Z$ is standard: coord are independent.

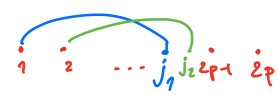
→ $\mathbb{E}[Z_1 Z_2] = \underbrace{\mathbb{E}[Z_1]}_{=0} \cdot \underbrace{\mathbb{E}[Z_2]}_{=0} = 0$

→ $\mathbb{E}[Z_1^2] = 1$

→ $\mathbb{E}[Z_1^4] = \mathbb{E}[\underbrace{Z_1 Z_1 Z_1 Z_1}_{i_1=i_2=i_3=i_4=1}] = \underbrace{1}_{\mathbb{E}[Z_1^2]\cdot\mathbb{E}[Z_1^2]} + 1 + 1 = 3$

→ $\mathbb{E}[Z_1^3] = 0$ because there are no pairings of 3 points

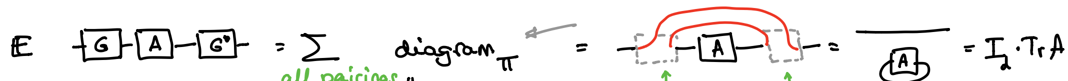
→ $\mathbb{E}[Z_1^{2p}] = \underbrace{1 + 1 + \cdots + 1}_{\#\text{ of pairings of } \{1,\dots,2p\}} = (2p-1)(2p-3)\cdots 5 \cdot 3 \cdot 1 = (2p)!!$



→ graphical Wick formula: want to compute

$\mathbb{E}\, G A G^*$, where $G$ = matrix $n \times k$ with indep. Gaussian entries (standard)

$A$ is a $k \times k$ fixed matrix



$= \sum_{\substack{\text{all pairings} \\ \text{of } G\text{-boxes with} \\ \bar{G} \text{ boxes}}} \text{diagram}_\pi = \ldots = I_? \cdot \mathrm{Tr} A$

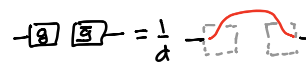
pairing

→ $\underset{\|x\|=1}{\mathbb{E}} |x\rangle\langle x|$ = average of all random pure states

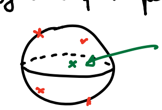
→ $|x\rangle = \dfrac{g}{\|g\|} \Rightarrow |x\rangle\langle x| = \dfrac{|g\rangle\langle g|}{\|g\|^2}$

→ since $|x\rangle$ and $\|g\|$ are independent $\mathbb{E}|g\rangle\langle g| = \mathbb{E}|x\rangle\langle x| \cdot \overbrace{\mathbb{E}[\|g\|^2]}^{d}$

$$\Rightarrow \mathbb{E}|x\rangle\langle x| = \frac{1}{d}\mathbb{E}|g\rangle\langle g| = \frac{1}{d}\mathbb{E} \; \ldots = \frac{1}{d} \; \ldots = \frac{1}{d} \cdot \ldots = I/d$$



→ the average of all pure quantum states is the maximally mixed state.

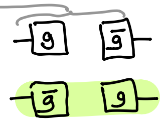
$d=2$ 

$I/2$

## Example for Isotropic states

$$T = \int_{\|x\|=1} \underbrace{|x\rangle\langle x| \otimes |\bar{x}\rangle\langle \bar{x}|}_{M_{d^2}} \, dx$$

→ $\int |g\rangle\langle g| \otimes |\bar{g}\rangle\langle \bar{g}| \; d\,\text{Gaussian}(g) = T \cdot \underbrace{\mathbb{E}\,\|g\|^4}_{d(d+1)}$

$\mathbb{E}$  $= \Sigma_{\text{all pairings of}}$ $g$'s with $\bar{g}$'s

→ $1^{st}$ pairing  $= I_{d^2}$

→ $2^{nd}$ pairing  $= d\,\omega_d$ ( maximally entangled state, un-normalized)

→ So: $T = \frac{1}{d(d+1)}\left[ I_{d^2} + d\,\omega_d \right] = \frac{1}{d+1}\,\omega_d + \left(1 - \frac{1}{d+1}\right)\frac{I}{d^2}$ (formula for isotropic states)