An entangled PPT state

- $\rho \in (M_{d_1} \otimes M_{d_2})^{1,+}$ quantum state
- $SEP(d_1 : d_2) = \{ \rho = \sum p_i \, |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i| \} = \{ \text{convex. comb. of product states} \}$

  $\rho \notin SEP$ is called entangled
- $PPT(d_1 : d_2) = \{ \rho^\Gamma = (id_{d_1} \otimes transp_{d_2})(\rho) \geq 0 \}$
- $SEP(d_1 : d_2) \subseteq PPT(d_1 : d_2)$ with equality $\iff d_1 \cdot d_2 \leq 6$
- we shall construct a $3 \otimes 3$ PPT entangled state $\rho \in PPT \setminus SEP$
- use unextendible product bases (UPB) : $X = \{x_1, \ldots, x_k\} \subseteq \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ such that:
  - they are orthonormal : $\langle x_i, x_j \rangle = \delta_{ij}$
  - they are product : $x_i = a_i \otimes b_i$ $\quad a_i \in \mathbb{C}^{d_1}, \; b_i \in \mathbb{C}^{d_2}$

    $\|a_i\| = \|b_i\| = 1$
  - the set X cannot be extended : $\nexists \, \alpha \in \mathbb{C}^{d_1}, \beta \in \mathbb{C}^{d_2}$ such that

    $\forall \, 1 \leq i \leq k \, , \; \langle \alpha \otimes \beta, x_i \rangle = 0$

Example 1 canonical basis in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$: $x_{ij} = |i\rangle \otimes |j\rangle$

It cannot be extended since it's the full basis

↳ we call it "trivial"

Example 2 A 5-element UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3$

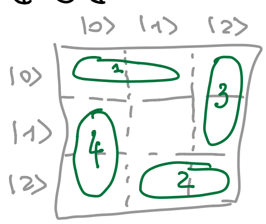
$$x_1 = \frac{1}{\sqrt{2}} |0\rangle \otimes (|0\rangle - |1\rangle)$$

$$x_2 = \frac{1}{\sqrt{2}} |2\rangle \otimes (|1\rangle - |2\rangle)$$

$$x_3 = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \otimes |2\rangle$$

$$x_4 = \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle) \otimes |0\rangle$$

$$x_5 = \frac{1}{3} (|0\rangle + |1\rangle + |2\rangle) \otimes (|0\rangle + |1\rangle + |2\rangle)$$



"tile states"

- $X = \{x_1, \ldots, x_5\}$ is an orthonormal family of product states
- Consider $\alpha \otimes \beta \perp x_i$ $\quad , i = 1 \ldots 5$ (⊛)
- $\langle \alpha \otimes \beta, x_1 \rangle = 0 \iff \langle \alpha, 0 \rangle = 0$ or $\langle \beta, |0\rangle - |1\rangle \rangle = 0$

→ ⓚ ⇒ α is $\perp$ to at least 3 elements of Y or β is $\perp$ to ≥ 3 elements of Y,

where $Y = \{ |0\rangle, |2\rangle, |0\rangle - |1\rangle, |1\rangle - |2\rangle, |0\rangle + |1\rangle + |2\rangle \}$

→ this is impossible, since any 3 elements of Y span $\mathbb{C}^3$ (can be checked by hand)

→ this shows that X is indeed an UPB.

Proposition Let $X = \{x_1, \ldots, x_k\} \subseteq \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ be a non-trivial ($k < d_1 d_2$) UPB. Then

$$\rho_X := \frac{1}{d_1 d_2 - k} \left(I_{d_1 d_2} - \sum_{i=1}^{k} |x_i\rangle\langle x_i| \right)$$ is a PPT entangled state.

Proof Actually ρ_X is the (normalized) projection on $\mathcal{X}^\perp$, where $\mathcal{X} = \operatorname{span}\{x_1, \ldots, x_k\}$

→ $$\rho_X^\Gamma = \frac{1}{d_1 d_2 - k} \Big(I_{d_1 d_2} - \sum \underbrace{|x_i\rangle\langle x_i|^\Gamma}_{|a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|} \Big) = \frac{1}{d_1 d_2 - k} \left(I - \sum_{i=1}^{k} |a_i\rangle\langle a_i| \otimes |\bar{b}_i\rangle\langle \bar{b}_i| \right) = \rho_Y$$

where $Y = \{ a_i \otimes \bar{b}_i \}_{i=1}^{k}$ is again an UPB ⇒ $\rho_X^\Gamma = \rho_Y \geq 0$ ⇒ ρ_X PPT

→ Assume $\rho_X \in SEP(d_1 : d_2)$ ⇒ $\rho_X = \sum_{j=1}^{r} p_j |\alpha_j\rangle\langle\alpha_j| \otimes |\beta_j\rangle\langle\beta_j|$ sep. decomp

⇒ $\forall\, 1 \leq j \leq r$, $\alpha_j \otimes \beta_j \in \mathcal{X}^\perp$ ⇒ $\alpha_1 \otimes \beta_1 \perp x_i \quad \forall\, 1 \leq i \leq k$

which contradicts X is an UPB ⇒ ρ_X entangled ∎

The DPS hierarchy

→ Doherty, Parillo, Spedalieri

→ Let $\rho \in SEP(d_1 : d_2)$ $\quad \rho_{AB} = \sum_{i=1}^{r} p_i \, |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|$

→ For all $k \geq 1$: $M_{d_1} \otimes M_{d_2}^{\otimes k} \ni \rho_{AB_1 \ldots B_k} \overset{(\diamond)}{:=} \sum_{i=1}^{r} p_i \, |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|^{\otimes k}$ is a quantum state s.t.

- symmetric under permutation of the Bobs.
- $\operatorname{Tr}_{B_2 B_3 \ldots B_k} \rho_{AB_1 \ldots B_k} = \rho_{AB}$

Definition $EXT_k(d_1 : d_2) := \{ \rho_{AB} \in M_{d_1 d_2}^{1,+} : \exists\, \sigma_{AB_1 \ldots B_k} \in M_{d_1 d_2^k}^{1,+}$ such that

σ is symmetric under perm. of the Bob systems and $\sigma_{AB_1} = \rho_{AB} \}$

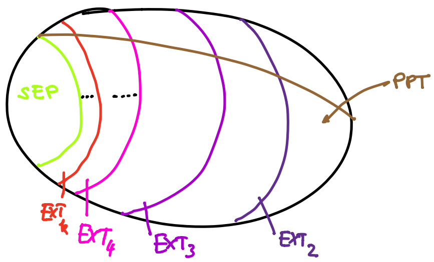
→ $SEP(d_1 : d_2) \subseteq EXT_k(d_1 : d_2) \quad \forall k \geq 1$ (see ($\diamond$))

→ if $k \leq \ell$ ⇒ $EXT_k(d_1 : d_2) \supseteq EXT_\ell(d_1 : d_2)$

↳ if $\rho \in EXT_\ell$ define $\sigma_{AB_1 \ldots B_k} := \operatorname{Tr}_{\ell - k}\, \sigma_{AB_1 \ldots B_\ell}$ is a k-ext of ρ

Theorem $\quad SEP(d_1 : d_2) = \bigcap_{k \geq 1} EXT_k(d_1 : d_2)$

→ in other words, a state ρ is entangled $\Leftrightarrow \exists\, k$ s.t. ρ does not have a k-sym. ext.



The DPS hierarchy in practice → Semidefinite Programs

→ linear programs (LP) : optimisation problems for which we have fast algorithms

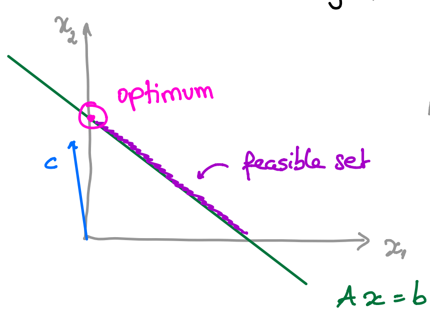
d variables ≥ 0
D constraints

$$\max \quad \langle C, x \rangle \qquad C \in \mathbb{R}^d$$

subject to: $\quad x_1, \dots, x_d \geq 0$

$$A x = b \quad \leftarrow \text{linear constraints in my variables}$$

$A \in M_{D \times d}$, $\quad b \in \mathbb{R}^D$



→ semidefinite programs (SDP)

Input: $A = A^\dagger \in M_d(\mathbb{C})$

$\Phi : M_d \to M_D$ preserves self-adjoint operators

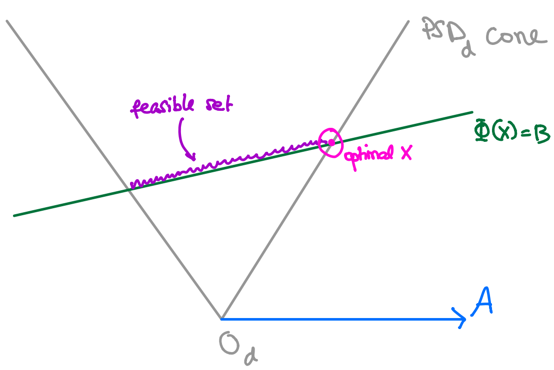
$B = B^\dagger \in M_D$

Variable $\quad X \in M_d(\mathbb{C})$

$$\max \quad \langle A, X \rangle$$

subject to: $\quad X \geq 0$ ← positive semidefinite

$$\Phi(X) = B$$



→ there exist poly-time algorithms to solve SDPs

Example 1 $\quad Z = Z^* \in M_d$. $\|Z\| = \max_{i=1}^{d} |\lambda_i|$ can be comp. with an SDP

eigs of Z

→ first try: $\max \lambda_i \leq t \iff Z \leq t \cdot I$

$$\|Z\| = \min \; t$$

← variable t $\quad t = \langle 1, t \rangle$

$$\text{subject to} \quad Z \leq tI$$

← $Z \leq tI \iff \Phi(t) \geq Z$ with $\Phi(t) = t \cdot I$

→ second try: $\|Z\| = \min \; t$

$$\text{subject to:} \quad -tI \leq Z \leq tI$$

Example 2 $\quad$ extendibility $\quad \mathrm{EXT}_k = \{ \rho_{AB} : \exists \, \sigma_{AB_1 \cdots B_k}$ symmetric in the B's such that $\sigma_{AB_i} = \rho_{AB} \}$

→ consider the SDP for a fixed ρ_{AB}

$$\alpha := \max \; \mathrm{Tr} \, \sigma_{AB_1 \cdots B_k}$$

subject to: $\sigma_{AB_1} = \rho_{AB}$

$\sigma_{AB_2} = \rho_{AB}$

$\vdots$

$\sigma_{AB_k} = \rho_{AB}$

$\sigma_{AB_1 \cdots B_k} \geq 0$

→ $\alpha \leq 1$, since $\mathrm{Tr}\, \sigma_{AB_1 \cdots B_k} = \mathrm{Tr}\, \sigma_{AB_1} = \mathrm{Tr}\, \rho_{AB} = 1 \Rightarrow$ if $\exists$ feasible point, $\alpha = 1$
if $\nexists$ feasible point, $\alpha = -\infty$

→ if $\rho_{AB} \in \mathrm{EXT}_k \Rightarrow \alpha = 1$

→ if $\alpha = 1 \Rightarrow \exists \; \sigma$ state with $\sigma_{AB_i} = \rho_{AB} \;\; \forall i$ $\quad$! σ might not be symmetric ...

↳ symmetrize σ: $\sigma' = \sum_{\pi \in S_k} \pi \cdot \sigma \cdot \pi^{-1}$ ← symmetrize the B's

Conclusion $\quad$ Testing $\rho_{AB} \in \mathrm{EXT}_k (d_1 : d_2)$ is an SDP with variable $\sigma \in M_{d_1 d_2^k}$

dim exponential in k