MULTI-PARTITE ENTANGLEMENT

→ bi-partite entanglement   Alice   Bob
$\mathcal{H}_A \otimes \mathcal{H}_B$
→ pure states
→ mixed state ent.

→ pure states $\mathcal{H}_A\otimes\mathcal{H}_B \ni |\psi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\,|a_i\rangle\otimes|b_i\rangle$ Schmidt decomposition

$|\psi\rangle$ separable $\Longleftrightarrow r=1 \Leftrightarrow |\psi\rangle = |a\rangle\otimes|b\rangle$

→ mixed states : $M_{d_1}\otimes M_{d_2} \ni \rho$ is separable $\Longleftrightarrow \rho = \sum_{i=1}^{m} p_i\, |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|$

→ multi-partite states : $\mathcal{H}_A\otimes\mathcal{H}_B\otimes\mathcal{H}_C\otimes\cdots$ ($\geqslant 3$ parties / factors in the $\otimes$)

→ focus on pure multi-partite entanglement (mixed case is very poorly understood)

→ $|\psi\rangle\in\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\cdots\otimes\mathbb{C}^{d_m}$ m-partite state $m\geqslant 3$

→ simplest case : $m=3$ $d_1=d_2=d_3=2$: 3 qubits

→ Examples of 3-partite states in $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$:

- $|000\rangle = |0\rangle\otimes|0\rangle\otimes|0\rangle$
- $|0\rangle_A \otimes \overbrace{\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)}^{|\Omega\rangle_{BC}}$
- $|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle+|111\rangle\right)$
- $|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle+|010\rangle+|100\rangle\right)$

Definition A state $|\psi\rangle\in\mathbb{C}^{d_1}\otimes\cdots\otimes\mathbb{C}^{m}$ is separable if $|\psi\rangle = |x_1\rangle\otimes|x_2\rangle\otimes\cdots\otimes|x_m\rangle$ for $x_i\in\mathbb{C}^{d_i}$ $i=1,\dots,m$.

Example $|000\rangle$, $|010\rangle$ are separable states.

Proposition $|\psi\rangle\in\mathbb{C}^{d_1}\otimes\cdots\otimes\mathbb{C}^{d_m}$ is separable if and only if $|\psi\rangle$ is bi-separable with respect to :

$\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2d_3\cdots d_m}$

$\mathbb{C}^{d_1d_2}\otimes\mathbb{C}^{d_3\cdots d_m}$

$\mathbb{C}^{d_1\cdots d_{m-1}}\otimes\mathbb{C}^{d_m}$

Proof Case $m=3$. $|\psi\rangle\in\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2d_3}$ is separable $\Rightarrow |\psi\rangle = |\alpha\rangle_A\otimes|\psi\rangle_{BC}$ (1)

$$|\psi\rangle \in \mathbb{C}^{d_1 d_2} \otimes \mathbb{C}^{d_3} \text{ is separable } \Rightarrow |\psi\rangle = |\psi\rangle_{AB} \otimes |\gamma\rangle_C \qquad (2)$$

→ $(2) \Rightarrow \mathrm{Tr}_A |\psi\rangle\langle\psi| = \mathrm{Tr}_A \left(|\psi_{A0}\rangle\langle\psi_{AB}| \otimes |\gamma\rangle\langle\gamma| \right) = \left[\mathrm{Tr}_A |\psi_{AB}\rangle\langle\psi_{AB}|\right] \otimes |\gamma\rangle\langle\gamma|$

→ $(1) \Rightarrow \mathrm{Tr}_A |\psi\rangle\langle\psi| = |\psi\rangle_{BC}\langle\psi_{BC}| \rightarrow |\psi_{BC}\rangle\langle\psi_{BC}| = \underbrace{\mathrm{Tr}_A |\psi_{AB}\rangle\langle\psi_{AB}|}_{\rho_B} \otimes |\gamma\rangle\langle\gamma|$

→ we have $|\psi\rangle\langle\psi| = |\alpha\rangle\langle\alpha| \otimes \rho_B \otimes |\gamma\rangle\langle\gamma|$

→ LHS is pure $\Rightarrow \rho_B$ pure $\Rightarrow \rho_B = |\beta\rangle\langle\beta| \Rightarrow |\psi\rangle = |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle$ separable ▨

Example GHZ state: $|GHZ\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^4$

$\mathrm{Tr}_{BC} |GHZ\rangle\langle GHZ| = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$ and $\mathrm{Tr}_C |GHZ\rangle\langle GHZ| = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$

↳ this is neither the max. ent. state on 2 qubits, nor the max. mixed state

→ we have $\mathrm{Tr}_C |GHZ\rangle\langle GHZ| = \frac{1}{2}\begin{array}{cccc|c} 00 & 01 & 10 & 11 & \\ 1 & & & & 00 \\ & 0 & & & 01 \\ & & & & 10 \\ & & & 1 & 11 \end{array}$ vs $\omega_2 = \frac{1}{2}\begin{bmatrix} 1 & & 1 \\ & 0 & \\ 1 & & 1 \end{bmatrix}$ vs $\frac{1}{4} I_4 = \frac{1}{4}\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$

TENSOR RANK

Definition The tensor rank of $x \in \mathbb{C}^{d_1} \otimes \dots \otimes \mathbb{C}^{d_m}$ is the smallest integer $R(x)$ such that

$$x = \sum_{i=1}^{R(x)} a_1^{(i)} \otimes a_2^{(i)} \otimes \dots \otimes a_m^{(i)}$$

→ a state $|\psi\rangle$ is separable $\Leftrightarrow R(|\psi\rangle) = 1$

→ $m=2$ tensor rank $\equiv$ Schmidt rank $\equiv$ usual rank for matrices

$$|\psi\rangle = \sum_{i=1}^{r} |\alpha_i\rangle \otimes |\beta_i\rangle \qquad X = \sum_{i=1}^{r} |y_i\rangle\langle z_i|$$

→ it is NP-hard to compute the tensor rank $(m \geq 3)$

→ Is there an SVD for tensors ? No because too many parameters in a tensor !

- → the Schmidt decomposition : $\underbrace{\mathbb{C}^d \otimes \mathbb{C}^d}_{2d^2} \ni |\psi\rangle = \sum_{i=1}^{d} \underset{\downarrow\, d}{\sqrt{\lambda_i}}\ \underbrace{|a_i\rangle}\overset{e^{i\varphi_i}}{\otimes} |b_i\rangle \rightarrow d^2$ real params.

  real parameters

  orthonormal vectors in $\mathbb{C}^d \rightarrow A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_d \\ | & | & & | \end{bmatrix}$ is unitary $\downarrow$ d^2 real params.

  in total, we have $d + d^2 + d^2 - d = 2d^2$

- → multipartite Schmidt decomposition ???

$$\underbrace{\mathbb{C}^d\otimes\mathbb{C}^d\otimes\mathbb{C}^d}_{2d^3} \ni |\psi\rangle = \underbrace{\sum_{i=1}^{d} \sqrt{\lambda_i}\; |a_i\rangle \otimes |b_i\rangle \otimes |c_i\rangle}_{\underset{\lambda}{d} \,+\, \underset{|a\rangle|b\rangle|c\rangle}{3d^2} \,-\, \underset{\text{counting phases multiple times}}{2d} \;=\; 3d^2 - d << 2d^3}$$

⇝ it cannot work, too few parameters on the RHS

Fact $\mathcal{U}_d = \{ UU^* = U^*U = I_d \}$ has d^2 real parameters:

d=2 $U = \begin{bmatrix} u_{11} & u_{21} \\ \underset{x}{\underbrace{u_{12}}} & \underset{y}{\underbrace{u_{22}}} \end{bmatrix} \equiv x, y \in \mathbb{C}^2 \quad \|x\| = \|y\| = 1$ and $x \perp y$

→ count parameters: $\underset{x}{\underbrace{2d}} + \underset{y}{\underbrace{2d}} - \underset{\mathbb{R} \ni \|x\|^2 = 1}{\underbrace{1}} - \underset{\|y\|=1}{\underbrace{1}} - \underset{\mathbb{C} \ni \langle x, y\rangle = 0}{\underbrace{2}} = 4 + 4 - 1 - 1 - 2 = 4 = 2^2$

Examples for tensor rank

$R(|GHZ\rangle) = 2$

→ $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ ⇝ 2-term decomposition ⇒ $R(GHZ) \leq 2$

→ $|GHZ\rangle$ is entangled, because $\mathrm{Tr}_{BC}\, |GHZ\rangle\langle GHZ| = \frac{1}{2} I$ and marginals of sep. states are pure
⇓
$R(GHZ) \geqslant 2$

Second example

$R(W) = 3$

→ $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \Rightarrow R(W) \leq 3$

→ W is entangled (compute $\mathrm{Tr}_{BC}\, |W\rangle\langle W|$ and see that it is not pure) ⇒ $R(W) \geqslant 2$

→ $R(W) = 2$ or 3 ??? Assume $R(W) = 2$ i.e. there is a decomposition

$$|W\rangle = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2$$

→ then, $\underset{M_2 \otimes M_2}{\underbrace{\mathrm{Tr}_C\, |W\rangle\langle W|}}$ contains at least two product states in its $\underset{\text{subspace} \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2}{\underbrace{\text{range}}}$

→ for W: $\mathrm{Tr}_C\, |W\rangle\langle W| = \mathrm{Tr}_C\big(|00\rangle\langle 00| \otimes |1\rangle\langle 1| + (|01\rangle + |10\rangle)(\langle 01| + \langle 10|) \otimes |0\rangle\langle 0| + \sim \otimes |0\rangle\langle 1| + \sim \otimes |1\rangle\langle 0| \big)$

$= |00\rangle\langle 00| + |01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|$

$\Rightarrow \mathrm{Ran}\, \mathrm{Tr}_C\, |W\rangle\langle W| = \mathbb{C}|00\rangle \oplus \mathbb{C}(|01\rangle + |10\rangle)$ ⇝ only contains 1 product state

→ So $R(W)$ cannot be 2 ⇒ $R(W) = 3$

→ one fundamental difference between $m = 2$ and $m \geqslant 3$ is that, for $m = 2$, the set

$A_r = \{ M_{d_1 \times d_2} \ni X$ such that $\mathrm{rk}(X) \leq r \}$ is closed: it is the intersection of finitely many closed sets of the form $\{ \det(X_{i_1 \cdots i_{r+1}; j_1 \cdots j_{r+1}} = 0) \}$ ↼ minors of order r+1 are zero

↳ this is not true for tensors: $\underbrace{\sqrt{3}|W\rangle}_{R=3} = \lim_{\varepsilon\to 0} \underbrace{\frac{(|0\rangle+\varepsilon|1\rangle)^{\otimes 3} - |000\rangle}{\varepsilon}}_{R=2}$

Def The border rank $\underline{R}(x)$ is the smallest integer s.t. $x = \lim_{n\to\infty} x_n$ with $R(x_n) \le \underline{R}(x)$

→ we have $\underline{R}(W)=2$ and $R(W)=3$

CLASSIFICATION OF ENTANGEMENT

→ consiter 2 states $|\varphi\rangle, |\psi\rangle \in \mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\cdots\otimes\mathbb{C}^{d_m}$. They are said to be

$|\varphi\rangle\sim|\psi\rangle$

SLOCC-equivalent if $\exists A_1\in GL_{d_1},\dots,A_m\in GL_{d_m}$ s.t. $|\varphi\rangle = A_1\otimes\cdots\otimes A_m|\psi\rangle$

↳ stochastic local operations and classical communication

can convert $|\psi\rangle\to|\varphi\rangle$ with prob > 0

$U_1\otimes U_2\otimes\cdots\otimes U_m$ local unitary (gates)

performing measurements + doing several rounds of the protocol

→ $\underline{m=2}$: $|\varphi\rangle\sim|\psi\rangle \iff$ Schmidt rank$(|\varphi\rangle)$ = Schmidt rank$(|\psi\rangle)$

↑ any dimension

$$\left.\begin{array}{l} |\varphi\rangle = \sum_{i=1}^{r}\sqrt{\lambda_i}\,|a_i\rangle\otimes|b_i\rangle \\ |\psi\rangle = \sum_{i=1}^{r}\sqrt{\mu_i}\,|\alpha_i\rangle\otimes|\beta_i\rangle \end{array}\right\} \Rightarrow A_1 = \sum_{i=1}^{r}\sqrt{\frac{\lambda_i}{\mu_i}}\,|a_i\rangle\langle\alpha_i|$$

$$A_2 = \text{—— " ——}\ |b_i\rangle\langle\beta_i|$$

→ so, in the case m=2, the rank is a complete invariant

→ for $m\geq 3$, the tensor rank is an invariant, but it is not complete.

Theorem There are 6 SLOCC equivalence classes for 3 qubits i.e. 3 qubits can be entangled in 6 different ways

R = 1 | $\vec r=(1,1,1)$ • separable states (A − B − C)

R=2 | • bi-separable states (A − BC), (B − AC), (C − AB)
$\vec r=(1,2,2)$ $\vec r=(2,1,2)$ $\vec r=(2,2,1)$

R=2 | $\vec r=(2,2,2)$ • GHZ class

R= 3 | $\vec r=(2,2,2)$ • W class

→ bi-separable : $|0\rangle_A\otimes\frac{1}{\sqrt2}(|00\rangle+|11\rangle)_{BC}$ belongs to the (A − BC)

→ to distinguish the 6 classes, use 2 invariants: R (the tensor rank) and $\vec r$

→ $\vec r = (r_A, r_B, r_C)$ where r_A = rank $\mathrm{Tr}_{BC}|\psi\rangle\langle\psi|$; r_B = rank $\mathrm{Tr}_{AC}|\psi\rangle\langle\psi|$, $r_c = \cdots$

→ $\vec{r}$ is an invariant: $|\varphi\rangle = A\otimes B\otimes C|\psi\rangle$ with A, B, C invertible $\Rightarrow$

$$|\varphi\rangle\langle\varphi| = A\otimes B\otimes C\,|\psi\rangle\langle\psi|\,A^\dagger\otimes B^\dagger\otimes C^\dagger \Rightarrow$$

$$\mathrm{Tr}_{BC}|\varphi\rangle\langle\varphi| = A\ \mathrm{Tr}_{BC}\left(I\otimes B\otimes C\,|\psi\rangle\langle\psi|\,I\otimes B^\dagger\otimes C^\dagger\right)A^\dagger$$

have the same rank since A, B, C invertible