APPROXIMATING GS OF LOC. GAPPED HAM. BY MPS (16/04/20)

→ Hilbert space $H = \bigotimes_{i=1}^{N} \mathbb{C}^d$; Hamiltonian $H = \sum_{i=1}^{N-1} H_{i,i+1}$ with $\| H_{i,i+1} \| \le J$

→ We assume that H is gapped : $\operatorname{spec} H = \{0\} \cup S$: $S \subseteq [\underbrace{\Delta E}_{\text{gap}}, +\infty)$

→ we say that $A \in B(H)$ has support in X if $A = A_X \otimes I_{X^c}$

Theorem [LR bound '72] $\exists\, C, \xi, v > 0$ s.t. $\forall A, B \in \mathcal{B}(H)$ such that $\operatorname{supp} A \subseteq X$, $\operatorname{supp} B \subseteq Y$

$$\| [A, B(t)] \| \le C \min(|X|, |Y|) \|A\| \|B\| \exp\left(2 \xi^{-1} (v|t| - d(X,Y))\right)$$

Corollary $\forall\, |t| \le \frac{d(X,Y)}{2v}$, $\| [A, B(t)] \| \le C \min(|X|, |Y|) \|A\| \|B\| \exp\left(- \frac{d(X,Y)}{\xi}\right)$

→ Hastings' idea : use this to prove the area law for the GS of a gapped Hamiltonian

Definition $\psi_0 \in \mathcal{H}$, $\|\psi_0\| = 1$, $H \psi_0 = 0$ $\quad \rho_{1,i} = \operatorname{Tr}_{i+1,\dots,N} |\psi_0 \rangle\langle \psi_0|$

Entanglement entropy $S(\rho_{1,i}) = - \operatorname{Tr} \rho_{1,i} \log \rho_{1,i}$

→ trivial bound : $S(\rho_{1,i}) \le i \log d$; when $S(\rho_{1,i}) \propto i$ → volume law

when $S(\rho_{1,i}) = O(1)$ → area law

Theorem [Hastings '07] $\exists\, S_{max} > 0$ s.t. $\max_i S(\rho_{1,i}) \le S_{max}$! Independent of N

Actually, $S_{max} = c_0 \xi' \ln \xi' \ln d \; 2^{\xi' \ln d}$ with $\xi' = 6 \min\left(\frac{2v}{\Delta E}, \xi\right)$

Today: use this entropy bound to $|\psi_0\rangle \approx$ MPS with "small" bond dimension

→ Rényi entropies $\alpha \ge 0$, $\alpha \ne 1$ $\quad S_\alpha(\rho) = \frac{1}{1-\alpha} \log \operatorname{Tr}(\rho^\alpha)$

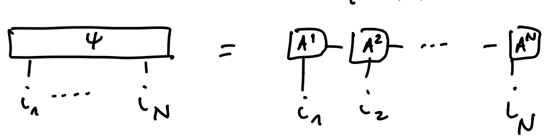
→ $S_1(\rho) := - \operatorname{Tr} \rho \log \rho = \lim_{\alpha \to 1} S_\alpha(\rho)$

→ $S_\infty(\rho) := - \log \|\rho\| \leftarrow \lambda_{max}(\rho)$

→ $S_0(\rho) = \log \dim \operatorname{supp}(\rho)$

→ Matrix Product state $(\mathbb{C}^d)^{\otimes N} \ni |\psi\rangle = \sum_{\underline{i}} \operatorname{Tr}[A^{[1]}_{i_1} \cdots A^{[N]}_{i_N}] \, |i_1 \cdots i_N\rangle$

here $A^{[i]} \in M_{D_i \times D'_{i+1}}$; $D'_{i-1} = D_i$; $D_1 = 1$; $D'_N = 1$



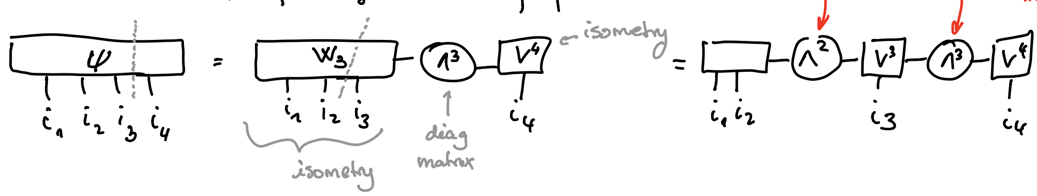
$D = \max D_i, D_i'$
= bond dimension

→ any state $|\psi\rangle$ can be written as a MPS with $D \sim d^{N/2}$ (exp bond dimension)

→ constant bond dimension ⇒ area law since $S(\rho_{1,j}) \leq \log D_j' \leq D$ ← want fixed bond dimension

→ How to obtain MPS form of an arbitrary q. state?

bond dimension = size of diag matrices



→ iterating SVD's we get MPS form with bond dimension at step i = rank ρ_{1i} = $\#\mathrm{supp}\,\Lambda^i$

→ how to obtain approx of $|\psi\rangle$ by MPS with small bond dimension?

Lemma 1 [MPS approximation with given bond dimension] [VC, Lemma 1]

Given a state $\psi \in (\mathbb{C}^d)^{\otimes N}$, construct usual MPS form with matrices Λ^j (large bond dim)

Define approx $|\psi_D\rangle$ by truncating Λ^j to their largest D singular values. Then

$$\|\psi - \psi_D\|^2 \leq 2\sum_{j=1}^{N-1} \varepsilon_j(D) \quad \text{where} \quad \varepsilon_j(D) = \underbrace{\sum_{i=D+1}^{N_j}}_{\text{sing values forgotten when truncating}} \Lambda^j(i)$$

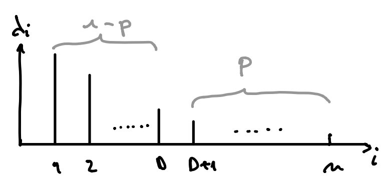
→ bound size of supp of Λ^j matrices ⇝ S_0 $\alpha = 0$ Rényi entropy

Lemma 2 Let λ be a prob. vector (singular values of Λ^j). $\varepsilon_D(\lambda) = \sum_{i \geq D+1} \lambda_i$ [VC, Lemma 2]
$\lambda_1 \geq \lambda_2 \geq \cdots$

Then $\forall\ 0 \leq \alpha < 1$
$$\log \varepsilon_D(\lambda) \leq \frac{1-\alpha}{\alpha} \underbrace{\left[S_\alpha(\lambda) - \log \frac{D}{1-\alpha} \right]}_{\text{want: } \leq 0}$$

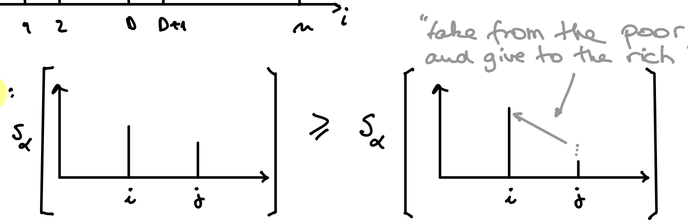
Proof idea Consider:

$\min S_\alpha(\lambda)$
s.t. $\sum_{i \geq D+1} \lambda_i = p$

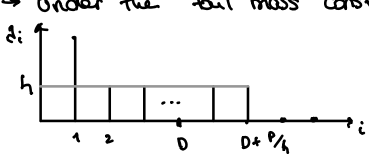


- Rényi entropies are Schur-concave:

$$p \prec q \Rightarrow S_\alpha(p) \geq S_\alpha(q)$$
↑ majorization



→ under the "tail mass" constraint $\sum_{i \geq D+1} \lambda_i = p$, the optimizer must be of the form



i.e. $\lambda_1 = 1 - p - (D-1)h$; $\lambda_2 = \cdots = \lambda_{D+p/h} = h$; $\lambda_{D+\frac{p}{h}+1} = \cdots = 0$

↳ optimize over h to get inequality ▨

Important remark

Hastings proves that the von Neumann entropy $S = S_{\alpha=1}$ is bounded, whereas the results above require $0 \leq \alpha < 1$. See [Has, Section 2] for the reduction in this case.

→ more general results about approx. by MPS under entropy bound are given in [SWVC]

[SWVC]

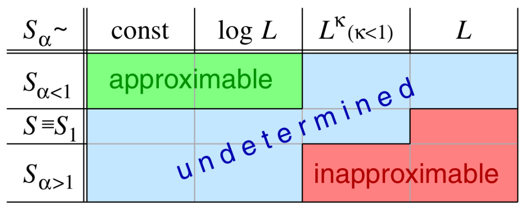


FIG. 1 (color online). Relation between scaling of block Rényi entropies and approximability by MPSs. In the ‘‘undetermined’’ region, nothing can be said about approximability just from looking at the scaling.

REFERENCES

[Has] Hastings - An area law for 1D q. syst (2007)

[VC] Verstraete, Cirac - MPS represent ground states faithfully

[SWVC] Schuch, Wolf, Verstraete, Cirac - Entropy scaling and simulability by MPS