Random quantum channels spectral properties & more

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Talk outline

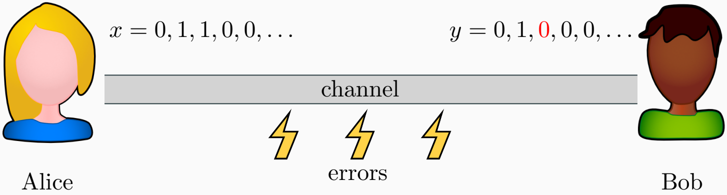
Random classical channels

Random quantum channels

Spectral gap of random quantum channels

Random classical channels

Classical channels



- Two parties, Alice and Bob want to communicate classically letters from the alphabet $\{1, 2, \ldots, d\}$
- Their communication channel is noisy:

$$\mathbb{P}[\text{Bob receives } j \mid \text{Alice sent } i] = M_{ij}$$

- Classical channels $\equiv$ Markov matrices acting on probability vectors
  - Positivity: for all i, j, $M_{ij} \geq 0$
  - Mass preservation: for all i, $\sum_j M_{ij} = 1$
- Example: bit flip channel $M = \begin{bmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{bmatrix}$

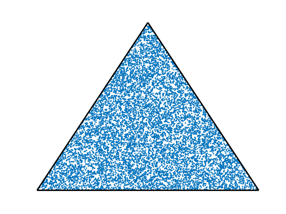
Random classical channels

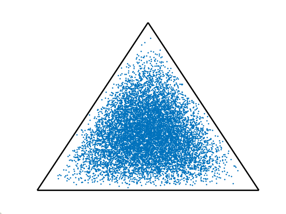
- Main idea: choose the rows of M i.i.d. from a given distribution μ on the probability simplex

$$\Delta_d := \{p \in \mathbb{R}^d \,:\, p_i \geq 0 \text{ and } \sum_i p_i = 1\}$$

- One standard choice is to use the Dirichlet distribution with parameter s (we write $p \sim \mathrm{Dir}_s(p_1, \ldots, p_d)$) if it has density proportional to

$$\mathrm{Dir}_s(p_1, \ldots, p_d) = p_1^{s-1} p_2^{s-1} \cdots p_{d-1}^{s-1} (\underbrace{1 - p_1 - \cdots - p_{d-1}}_{p_d})^{s-1}.$$





Dirichlet distributions (10^5 samples) on Δ_3, for $s = 1$ (left, uniform distribution on the simplex) and $s = 3$ (right)

Random classical channels - spectrum and gap

The behavior of the spectrum of a random Markov map has been studied by Bordenave, Caputo, Chafai - *Circular law theorem for random Markov matrices* - PTRF 2012. They show that if the rows of M are obtained by normalizing an i.i.d. vector with entries X_{ij} with $\sigma^2 = \mathrm{Var}(X_{ij})$, then

Theorem 1.1 (Quartercircular law theorem) *We have a.s.*

$$\nu_{\sqrt{n}M} \xrightarrow[n\to\infty]{\mathscr{C}_b} \mathcal{Q}_\sigma .$$

Theorem 1.2 (Extremes) *We have $\lambda_1(M) = 1$. Moreover, if $\mathbb{E}(|X_{1,1}|^4) < \infty$ then a.s.*

$$\lim_{n\to\infty} s_1(M) = 1 \quad \textit{and} \quad \lim_{n\to\infty} s_2(\sqrt{n}M) = 2\sigma \quad \textit{while} \quad \varlimsup_{n\to\infty} |\lambda_2(\sqrt{n}M)| \le 2\sigma.$$

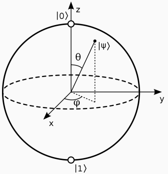
Theorem 1.3 (Circular law theorem) *If $X_{1,1}$ has a bounded density then a.s.*

$$\mu_{\sqrt{n}M} \xrightarrow[n\to\infty]{\mathscr{C}_b} \mathcal{U}_\sigma .$$

Random quantum channels

Quantum mechanics on one slide

- Pure quantum states of one particle: unit norm vectors inside a complex Hilbert space $\mathcal{H} = \mathbb{C}^d$ [*classical states:* $x \in \{1, 2, \ldots, d\}$]
- Mixed quantum states (or density matrices): positive semidefinite matrices of unit trace $\rho \geq 0$, $\operatorname{Tr} \rho = 1$ [*classical mixed states:* $p \in \Delta_d$ *probability distribution*]. Importantly, the set of quantum states is **not** a simplex. Below, the situation for $d = 2$, segment vs. Bloch ball:



- Extreme points of the set of mixed states: $P_x = |x\rangle\langle x|$, with $x \in \mathbb{C}^d$, $\|x\| = 1$ [*extreme classical mixed state:* $p = \delta_x$ *for* $x \in \{1, 2, \ldots, d\}$]
- More particles $\rightsquigarrow$ take the tensor product of the Hilbert spaces [*classical states:* $\{1, 2, \ldots, d_1\} \times \{1, 2, \ldots, d_2\}$]
- Quantum marginal: partial trace operation $\rho^{(1)} := [\mathrm{id} \otimes \operatorname{Tr}](\rho^{(12)})$ [*classical marginal:* $p_i^{(1)} = \sum_j p_{ij}^{(12)}$]

Quantum channels

Channels	Deterministic	Noisy
Classical	$f : [d] \to [d]$	M Markov: $M_{ij} \geq 0$ and $\forall i$, $\sum_j M_{ij} = 1$
Quantum	$U \in \mathcal{U}(d)$	Φ completely positive, trace pres. map

- Classical channels (acting on probability vectors):
  - Positivity: for all i, j, $M_{ij} \geq 0$
  - Mass preservation: for all j, $\sum_i M_{ij} = 1$.
- Quantum channels: CPTP linear maps $\Phi : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$
  - CP - complete positivity: $\Phi \otimes \mathrm{id}_k$ is a positive map, $\forall k \geq 1$. Positivity:

$$X \text{ positive semi-definite} \implies \Psi(X) \text{ positive semi-definite}$$

  - TP - trace preservation: $\mathrm{Tr} \circ \Phi = \mathrm{Tr}$.

Structure of quantum channels

Theorem [Stinespring-Kraus-Choi]

Let $\Phi : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ be a linear map. TFAE:

1. The map Φ is completely positive and trace preserving (CPTP).
2. [Stinespring] There exist an integer s ($s = d_1 d_2$ suffices) and an isometry $W : \mathbb{C}^{d_1} \to \mathbb{C}^{d_2} \otimes \mathbb{C}^{s}$ such that
$$\Phi(X) = [\mathrm{id}_{d_2} \otimes \mathrm{Tr}_s](WXW^*).$$
3. [Kraus] There exist operators $A_1, \ldots, A_s \in \mathcal{M}_{d_2 \times d_1}$ satisfying $\sum_i A_i^* A_i = I_{d_1}$ such that
$$\Phi(X) = \sum_{i=1}^{s} A_i X A_i^*.$$
4. [Choi] The Choi matrix C_Φ is positive semidefinite, where
$$C_\Phi := \sum_{i,j=1}^{d_1} E_{ij} \otimes \Phi(E_{ij}) \in \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2}$$
and $[\mathrm{id}_{d_1} \otimes \mathrm{Tr}_{d_2}](C_\Phi) = I_{d_1}$.

Examples and non-examples

- The identity channel $\mathrm{id} : \mathcal{M}_d \to \mathcal{M}_d$ has the (un-normalized) Bell state as its Choi matrix

$$C_{\mathrm{id}} = \sum_{i,j=1}^{d} |ii\rangle\langle jj| = \sum_{i,j=1}^{d} e_i \otimes e_i \cdot e_j^* \otimes e_j^*.$$

- The totally depolarizing channel (or the conditional expectation on scalars) $\Delta(X) = (\operatorname{Tr} X) I/d$ has Choi matrix I_{d^2}/d
- The totally dephasing channel (or the conditional expectation on diagonal matrices) D has Kraus decomposition

$$D(\rho) = \sum_{i=1}^{d} |i\rangle\langle i|\rho|i\rangle\langle i|.$$

- The transposition $\Theta(\rho) = \rho^\top$ is **not** a quantum channel, since it is not completely positive. Its Choi matrix is $C_\Theta = F$, where F is the flip operator $Fx \otimes y = y \otimes x$. F has eigenvalues $+1$ with multiplicity $d(d+1)/2$ and -1 with multiplicity $d(d-1)/2$.

Random quantum channels

There exist several natural candidates for probability distributions on the set of quantum channels $\{\Phi : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}\}$

1. The Lebesgue measure: the set of quantum channels is convex and compact, having real dimension $d_1^2 d_2^2 - d_1^2$. Normalize the volume measure to obtain a probability distribution $\mu_{d_1,d_2}^{Lebesgue}$
2. Pick the isometry W in the Stinespring decomposition at random: W is a Haar-random isometry $\mathbb{C}^{d_1} \to \mathbb{C}^{d_2} \otimes \mathbb{C}^s$. We obtain a probability distribution $\mu_{d_1,d_2;s}^{Stinespring}$, where $s \geq 1$ is an integer such that $d_1 \leq sd_2$
3. Pick the Kraus operators A_i at random: G_i are i.i.d. $d_2 \times d_1$ Ginibre matrices, define $A_i = G_i S^{-1/2}$, with $S = \sum_{i=1}^s G_i^* G_i$. We obtain a probability distribution $\mu_{d_1,d_2;s}^{Kraus}$, where $s \geq 1$ is an integer such that $d_1 \leq sd_2$
4. Pick the Choi matrix at random: $\tilde{C}$ is a Wishart matrix of parameters $d_1 d_2, s)$, define $C := [I \otimes T^{-1/2}]\tilde{C}[I \otimes T^{-1/2}]^*$, with $T = [\mathrm{Tr} \otimes \mathrm{id}]\tilde{C}$. We obtain a probability distribution $\mu_{d_1,d_2;s}^{Choi}$, where $s \geq 1$ is any real number $s \geq d_1 d_2$, or an integer $s \geq d_1/d_2$

Equivalence of probability measures

Theorem (Kukulski, N., Pawela, Puchala, Zyczkowski '20)

The above distributions are identical, when the respective parameters match:

$$\mu^{\text{Lebesgue}}_{d_1,d_2} \in \left\{ \mu^{\text{Stinespring}}_{d_1,d_2;s} \right\}_{\substack{s\in\mathbb{N} \\ s\geq d_2/d_1}} = \left\{ \mu^{\text{Kraus}}_{d_1,d_2;s} \right\}_{\substack{s\in\mathbb{N} \\ s\geq d_2/d_1}} \subset \left\{ \mu^{\text{Choi}}_{d_1,d_2;s} \right\}_{s\in\mathcal{S}_{d_1,d_2}}$$

where

$$\mathcal{S}_{d_1,d_2} := \left\{ \left\lceil \frac{d_1}{d_2} \right\rceil, \left\lceil \frac{d_1}{d_2} \right\rceil + 1, \ldots, d_1 d_2 - 1 \right\} \sqcup [d_1 d_2, +\infty)$$

The Lebesgue measure is obtained for $s = d_1 d_2$.

Computationally, the random Kraus operators procedure is the cheapest; mathematically, the random isometry procedure is the more interesting and easier to deal with, since no normalization procedure is needed, and the structure of Haar random isometry is well understood

More on the distribution of random quantum channels

- The density of the normalized Choi matrix reads

$$f(C) = \delta([\mathrm{id} \otimes \mathrm{Tr}](C) - I_{d_1}) \det C^{s-d_1 d_2} \mathrm{dLeb}$$

- For any fixed pure state $P_x = xx^*$, the output matrix $\rho = \Phi(P_x)$ follows the induced distribution of parameters (d_2, s), i.e. has the distribution of a trace-normalized random Wishart matrix
- However, different inputs yield correlated outputs! It is an interesting problem to study the distribution of the random output set

$$\Phi(\{\rho \in \mathcal{M}_{d_1} \; : \; \rho \geq 0 \text{ and } \mathrm{Tr}\, \rho = 1\})$$

- Open question: what are the properties of the Lebesgue distribution on the set of unital quantum channels $\Phi(I) = I$? In the classical case (bistochastic matrices) the problem has been studied by Chatterjee, Diaconis, Sly - *Properties of uniform doubly stochastic matrices* - arXiv:1010.6136

Spectral gap of random quantum channels

Super-operators

- Given a quantum channel $\Phi : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$, consider its super-operator F, which is the matrix of Φ seen as a linear operator $\Phi : \mathbb{C}^{d_1^2} \to \mathbb{C}^{d_2^2}$

$$F = \sum_{i=1}^{s} A_i \otimes \overline{A_i} \in \mathcal{M}_{d_2^2 \times d_1^2}$$

- It is the matrix F which is analogous to the Markov matrix M of a classical channel. Note that F is not self-adjoint (nor positive) in general

Theorem (Quantum Perron-Frobenius)

Let $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ be a positive map with spectral radius r. Then r is an eigenvalue of F and there is a positive semi-definite matrix $X \in \mathcal{M}_d$ such that $\Phi(X) = rX$.

- For quantum channels, the spectral radius is $r = 1$
- We shall be interested in the spectral gap: assuming Φ has an unique fixed point

$$\mathrm{gap}(\Phi) = 1 - \max_{\lambda \in \mathrm{spec}\, F,\ \lambda \neq 1} |\lambda|$$

Spectrum of the super-operator

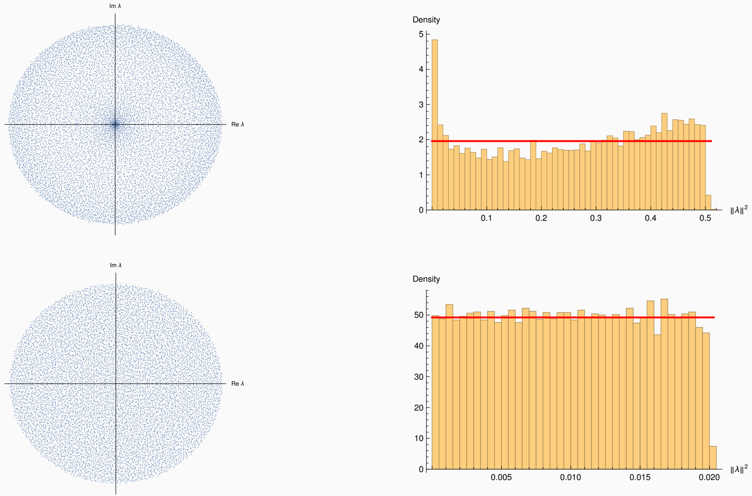


Figure 1: Eigenvalues of the superoperators of random quantum channels: single sample of a random quantum channel $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ with $d = 100$. The parameter s is, respectively, 2 and 50 for the top and bottom rows

Main result

- We shall work in the quantum expander regime:

$$s \text{ fixed}, \quad d_2 = d \to \infty, \quad d_1 \sim \gamma d \to \infty, \quad \gamma \in (0, s) \text{ fixed}$$

Theorem (Gonzalez-Guillen, Junge, N. '18, arXiv:1811.08847)

Consider a sequence of random quantum channels $\Phi_d : \mathcal{M}_d \to \mathcal{M}_d$ (we assume here $\gamma = 1$) and let F_d be the corresponding super-operator sequence. Then, almost surely as $d \to \infty$, the second largest (in absolute value) eigenvalue of F_d is asymptotically upper bounded:

$$\limsup_{d\to\infty} |\lambda_2(F_d)| \leq \left(\sqrt{1 + \frac{s-1}{s^2}} + g_{s,1} \right) g_{s,1}$$

In particular, we have the following asymptotic (in s) lower bound for the spectral gap:

$$\liminf_{d\to\infty} 1 - |\lambda_2(F_d)| \gtrsim 1 - \frac{8}{\sqrt{s}}$$

- See also Cécilia Lancien, David Pérez-García - *Correlation length in random MPS and PEPS* - arXiv:1906.11682

Proof strategy - 3 steps

1. Eigen vs. singular values: Weyl's Majorant Theorem

$$\forall p > 0 \qquad 1 + |\lambda_2(F)|^p \leq s_1(F)^p + s_2(F)^p$$

2. Lower bound on the largest singular value of F

Theorem (uses Weingarten calculus)

Consider a sequence of random quantum channels $\Phi : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ and let F be the corresponding super-operator. Define the overlap

$$\mathbb{R} \ni f := \mathrm{Tr}[\omega_{d_1} \cdot F^* F],$$

where ω_{d_1} is the maximally entangled quantum state

$$\omega_{d_1} = \frac{1}{d_1} \sum_{i,j=1}^{d_1} |ii\rangle\langle jj|$$

Then, for all integers $p \geq 1$

$$\lim_{d_{1,2}\to\infty} \mathbb{E} f^p = \left(\gamma + \frac{1}{s} - \frac{\gamma}{s^2}\right)^p .$$

3. Upper bound on the norm of the restriction $\rightsquigarrow \cdots$

The upper bound

- We have guessed that the maximally entangled vector

$$\Omega_{d_1} = \frac{1}{\sqrt{d_1}} \sum_{i=1}^{d_1} |ii\rangle$$

  is close to the Perron-Frobenius (right) eigenvector of F
- We want now to upper bound the norm of the restriction $F(I_{d_1^2} - \omega_{d_1})$
- We use ideas from
  - Hastings - *Random unitaries give quantum expanders* - PRA 2007
  - Pisier - *Quantum expanders and geometry of operator spaces* - JEMS 2014
- Decoupling: F is defined via a Haar-isometry W $\rightsquigarrow$ decouple the s blocks of V to i.i.d. Ginibre matrices Y_i $\rightsquigarrow$ decouple $\sum_i (Y_i \otimes \bar{Y}_i)(I - \omega)$ to $\sum_i Y_i \otimes Z_i$
- The isometry W can be obtained from a Ginibre random matrix Y by its polar decomposition $Y = W|Y|$. If $\mathcal{E}$ is the conditional expectation on the σ-algebra generated by W, we have

$$\mathcal{E}(Y \otimes \overline{Y}) = \mathcal{E}(W|Y| \otimes \overline{W|Y|}) = (W \otimes \overline{W})\mathbb{E}(|Y| \otimes |\overline{Y}|).$$

The upper bound

$$\mathcal{E}(Y \otimes \overline{Y}) = \mathcal{E}(W|Y| \otimes \overline{W|Y|}) = (W \otimes \overline{W})\mathbb{E}(|Y| \otimes |\overline{Y}|).$$

One can compute

$$\mathbb{E}(|Y| \otimes |\overline{Y}|) = \left[\omega_{d_1} + \chi_{d_2 s, d_1}(I_{d_1^2} - \omega_{d_1})\right],$$

where

$$\chi_{M,N} := \frac{\mathbb{E}\|Y\|_1^2 - 1}{N^2 - 1}.$$

For all M, N, we have $\chi_{M,N} \geq 1/(N+1) > 0$. Moreover, in the limit where $N \to \infty$ and $M \sim cN$ for some constant $c \geq 1$,

$$\lim_{N \to \infty} \chi_{cN,N} = \chi_c := c^{-1} \left[\int_a^b \frac{\sqrt{(x-a)(b-x)}}{2\pi\sqrt{x}} \mathrm{d}x\right]^2,$$

where $a = (\sqrt{c} - 1)^2$ and $b = (\sqrt{c} + 1)^2$.

The upper bound

Write $Y = \sum_{i=1}^{s} Y_i \otimes |i\rangle$.

Theorem

Let $Y_1, \ldots, Y_s$ be independent $d_2 \times d_1$ Ginibre matrices, and consider independent copies $Z_1, \ldots, Z_s$ having the same distributions. Then, for all $p \geq 1$ and all $1 \leq q \leq \infty$, we have

$$\mathbb{E}\left\|\sum_{i=1}^{s}(Y_i \otimes \overline{Y_i})(I_{d_1^2} - \omega_{d_1})\right\|_q^p \leq 2^p \mathbb{E}\left\|\sum_{i=1}^{s}(Y_i \otimes Z_i)(I_{d_1^2} - \omega_{d_1})\right\|_q^p$$

Theorem

Let $Y_1, \ldots, Y_s, Z_1, \ldots, Z_s$ be independent Ginibre random matrices of parameters $(d_2, d_1; (d_2 s)^{-1})$. Then, for all even integers $p \geq 2$,

$$\mathbb{E}\left\|\sum_{i=1}^{s} Y_i \otimes Z_i\right\|_\infty^p \leq d_2^2 \left(\frac{(1+\sqrt{\gamma})^2}{\sqrt{s}} + \varepsilon + \beta\sqrt{\frac{p}{d_2}}\right)^p$$

where $\varepsilon \to 0$ and β is bounded, as $d_{1,2} \to \infty$.

The take-home slide

Channels	Deterministic	Noisy
Classical	$f : [d] \to [d]$	M Markov: $M_{ij} \geq 0$ and $\forall i, \sum_j M_{ij} = 1$
Quantum	$U \in \mathcal{U}(d)$	Φ completely positive, trace pres. map

- Random quantum channels: equivalent definitions
  1. The Lebesgue measure: normalize the volume measure
  2. Stinespring dilation: $\Phi(\rho) = [\mathrm{id} \otimes \mathrm{Tr}](W \rho W^*)$ for a Haar-random isometry $W : \mathbb{C}^{d_1} \to \mathbb{C}^{d_2} \otimes \mathbb{C}^s$
  3. Kraus decomposition: $\Phi(\rho) = \sum_i A_i \rho A_i^*$ with A_i random normalized Ginibre matrices ($A_i = G_i S^{-1/2}$, with $S = \sum_{i=1}^s G_i^* G_i$)
  4. Random Choi matrix: $\tilde{C}$ is a Wishart $(d_1 d_2, s)$ random matrix and $C = [I \otimes T^{-1/2}]\tilde{C}[I \otimes T^{-1/2}]^*$, with $T = [\mathrm{Tr} \otimes \mathrm{id}]\tilde{C}$
- The Lebesgue measure corresponds to $s = d_1 d_2$
- Spectral gap: almost surely, as $1 \ll s \ll d$

$$\liminf_{d \to \infty} 1 - |\lambda_2(F_d)| \gtrsim 1 - \frac{8}{\sqrt{s}}$$