The PPT^2 conjecture holds for diagonal unitary covariant maps

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Talk outline

The PPT^2 conjecture

Diagonal unitary covariant maps

Pairwise completely positive matrices and factor width

PPT^2 holds for (C)DUC maps

The PPT^2 conjecture

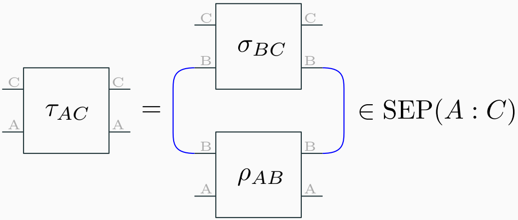
Statement of the conjecture

- A linear map $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ is said to be PPT if both Φ and $\Phi \circ \top$ are completely positive

Conjecture ([Chr12])

For any PPT map Φ, the composition $\Phi \circ \Phi$ is entanglement breaking

- At the level of Choi matrices, the conjecture is equivalent to: $\rho_{AB} \in \mathrm{PPT}(A : B)$ and $\sigma_{BC} \in \mathrm{PPT}(B : C)$ implies



Progress on the proof

- Trivially holds for qubits since $\mathrm{PPT} \iff \mathrm{SEP}$ for $d = 2$
- The distance between iterates of a unital (or trace preserving) PPT map and the set of entanglement breaking maps tends to zero in the asymptotic limit [KMP18]
- Any unital (or trace preserving) PPT map becomes entanglement breaking after finitely many iterations of composition with itself [RJP18]
- For other algebraic approaches and extensions, see [LG15, HRF20, GKS20]
- The conjecture holds for fully unitary covariant channels [VW01, CMHW19]
- Independent random quantum channels satisfy the conjecture [CYZ18]
- Gaussian maps satisfy the conjecture [CMHW19]
- The conjecture holds for qutrits [CYT19, CMHW19]

Theorem ([SN20b])

PPT^2 holds for diagonal unitary covariant maps

Diagonal unitary covariant maps

Main definition

- Consider the set $\mathcal{DU}_d$ of diagonal unitary operators

$$U = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_d}), \qquad \theta_j \in \mathbb{R}$$

Definition

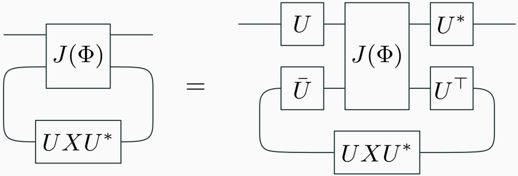
A linear map $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ is said to be

- conjugate diagonal unitary covariant (CDUC) if $\forall X \in \mathcal{M}_d, \forall U \in \mathcal{DU}_d$

$$\Phi(UXU^*) = U\Phi(X)U^* \iff (U \otimes \bar{U})J(\Phi)(U \otimes \bar{U})^* = J(\Phi)$$

- diagonal unitary covariant (DUC) if $\forall X \in \mathcal{M}_d, \forall U \in \mathcal{DU}_d$

$$\Phi(UXU^*) = U^*\Phi(X)U \iff (U \otimes U)J(\Phi)(U \otimes U)^* = J(\Phi)$$



Explicit form

Proposition ([SN20a])

The vector space of CDUC (resp. DUC) maps is parametrized by pairs of matrices $(A, B) \in \mathcal{M}_d^2$ *having the same diagonal* $\operatorname{diag} A = \operatorname{diag} B$*:*

$$\Phi_{A,B}(X) = \operatorname{diag}(A\,|\operatorname{diag} X\rangle) + (B - \operatorname{diag} B) \odot X$$

$$\Psi_{A,C}(X) = \operatorname{diag}(A\,|\operatorname{diag} X\rangle) + (C - \operatorname{diag} C) \odot X^\top$$

- The identity channel is CDUC: $A = I$, $B = J$
- The completely depolarizing channel is (C)DUC: $A = J$, $B/C = I$
- The transposition map is DUC: $A = I$, $C = J$
- The Choi map $\Phi_{\mathrm{Choi}} : \mathcal{M}_3 \to \mathcal{M}_3$ is CDUC

$$\Phi_{\mathrm{Choi}}(X) = \begin{bmatrix} X_{11} + X_{33} & -X_{12} & -X_{13} \\ -X_{21} & X_{11} + X_{22} & -X_{23} \\ -X_{31} & -X_{32} & X_{22} + X_{33} \end{bmatrix}$$

with $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

Properties of (C)DUC maps

- The positivity properties of (C)DUC maps depend on the convex cones

$$\mathrm{EWP}_d = \{A \in \mathcal{M}_d \,:\, A_{ij} \geq 0 \quad \forall i,j\}$$

$$\mathrm{PSD}_d = \{B \in \mathcal{M}_d \,:\, B \text{ is positive semidefinite, i.e. } B = ZZ^*\}$$

Proposition ([SN20a])

Let $A, B \in \mathcal{M}_d$ with $\operatorname{diag} A = \operatorname{diag} B$. Then

- $\Phi_{A,B}$ *is CP* $\iff$ $\Psi_{A,B}$ *is coCP* $\iff$ $A \in \mathrm{EWP}_d$ *and* $B \in \mathrm{PSD}_d$
- $\Phi_{A,B}$ *is coCP* $\iff$ $\Psi_{A,B}$ *is CP* $\iff$
  $A \in \mathrm{EWP}_d$ *and* $A_{ij}A_{ji} \geq |B_{ij}|^2 \quad \forall i,j$
- $\Phi_{A,B}$ *is PPT* $\iff$ $\Psi_{A,B}$ *is PPT* $\iff$
  $A \in \mathrm{EWP}_d$ *and* $B \in \mathrm{PSD}_d$ *and* $A_{ij}A_{ji} \geq |B_{ij}|^2 \quad \forall i,j$
- $\Phi_{A,B}$ *is unital* $\iff$ $\Psi_{A,B}$ *is unital* $\iff$ $\sum_j A_{ij} = 1 \quad \forall i$
- $\Phi_{A,B}$ *is trace pres.* $\iff$ $\Psi_{A,B}$ *is trace pres.* $\iff$ $\sum_i A_{ij} = 1 \quad \forall j$

$\rightsquigarrow$ CDUC channels are mixtures of classical channels and Schur multipliers

Pairwise completely positive matrices and factor width

Entanglement breaking (C)DUC maps and the PCP cone

Proposition ([SN20a])

Let $A, B \in \mathcal{M}_d$ *with* $\operatorname{diag} A = \operatorname{diag} B$. *Then*

$$\Phi_{A,B} \text{ is EB} \iff \Psi_{A,B} \text{ is EB} \iff (A,B) \in \mathrm{PCP}_d$$

Definition (Johnston and MacLean [JM19])

A pair $(A, B) \in \mathcal{M}_d^2$ is said to be pairwise completely positive (PCP) if there exist finitely many vectors $v_n, w_n \in \mathbb{C}^d$ such that

$$A = \sum_n |v_n \odot \overline{v_n}\rangle\langle w_n \odot \overline{w_n}| \qquad B = \sum_n |v_n \odot w_n\rangle\langle v_n \odot w_n|$$

- A pair (A, A) is PCP iff A is completely positive [BSM03]:

$$A = ZZ^\top \quad \textit{for} \quad Z \text{ with } Z_{ij} \geq 0 \; \forall i,j \qquad (\implies A \in \mathrm{EWP}_d \cap \mathrm{PSD}_d)$$

- Deciding membership in PCP_d is NP-hard

Factor width

Definition ([BCPT05])

A matrix $B \in \mathrm{PSD}_d$ is said to have factor width k if it admits a rank one decomposition $B = \sum_n |z_n\rangle\langle z_n|$, such that, for all n, $\#\,\mathrm{supp}(z_n) \leq k$

Definition ([SN20b])

A matrix pair $(A, B) \in \mathrm{PCP}_d$ is said to have factor width k if it admits a *PCP* decomposition $A = \sum_n |v_n \odot \overline{v_n}\rangle\langle w_n \odot \overline{w_n}|$ and $B = \sum_n |v_n \odot w_n\rangle\langle v_n \odot w_n|$ with $\#\,\mathrm{supp}(v_n \odot w_n) \leq k$ for all n

- The sets above are denoted by PSD_d^k, resp. PCP_d^k
- We have the following inclusions

$$\mathrm{PSD}_d^1 \subseteq \mathrm{PSD}_d^2 \subseteq \cdots \subseteq \mathrm{PSD}_d^d = \mathrm{PSD}_d$$
$$\mathrm{PCP}_d^1 \subseteq \mathrm{PCP}_d^2 \subseteq \cdots \subseteq \mathrm{PCP}_d^d = \mathrm{PCP}_d$$

- PSD_d^1 is the set of diagonal matrices in EWP_d
- PCP_d^1 is the set of matrix pairs $(A, B) \in \mathrm{PCP}_d$ such that $B = \operatorname{diag} A$

Factor width two

- To any matrix B, associate its comparison matrix

$$M(B)_{ij} = \begin{cases} |B_{ij}| & \text{if } i = j \\ -|B_{ij}| & \text{otherwise} \end{cases}$$

Proposition ([BCPT05])

For a (hermitian) matrix $B \in \mathcal{M}_d$, the following equivalences hold:

$$B \in \mathrm{PSD}_d^2 \iff M(B) \in \mathrm{PSD}_d \iff B \text{ is scaled diagonally dominant}$$

Proposition ([SN20b])

For $A, B \in \mathcal{M}_d$ such that

$$A \in \mathrm{EWP}_d \quad \text{and} \quad B \in \mathrm{PSD}_d \quad \text{and} \quad A_{ij}A_{ji} \geq |B_{ij}|^2 \quad \forall i, j$$

the following equivalence holds:

$$(A, B) \in \mathrm{PCP}_d^2 \iff B \in \mathrm{PSD}_d^2 \iff M(B) \in \mathrm{PSD}_d$$

$\rightsquigarrow$ A simple criterion for membership inside $\mathrm{PCP}_d^2 \subseteq \mathrm{PCP}_d (\leftrightarrow \text{EB})$

PPT^2 holds for (C)DUC maps

Composition of (C)DUC maps

- Let Φ denote a CDUC map and Ψ denote a DUC map
- We have, for all $A, B, C, D \in \mathcal{M}_d$,

$$\Phi_{A,B} \circ \Phi_{C,D} = \Phi_{X,Y}$$
$$\Psi_{A,B} \circ \Psi_{C,D} = \Phi_{Z,W}$$
$$\Phi_{A,B} \circ \Psi_{C,D} = \Psi_{X,Y}$$
$$\Psi_{A,B} \circ \Phi_{C,D} = \Psi_{Z,W}$$

with

$$(X, Y) := (AC, B \odot D + \mathrm{diag}(AC - B \odot D))$$
$$(Z, W) := (AC, B \odot D^\top + \mathrm{diag}(AC - B \odot D^\top))$$

The main result

Theorem ([SN20b])

Let $\Phi_{A,B}$, $\Phi_{C,D}$ be two PPT (C)DUC maps, and let (X, Y) be the matrix pair corresponding to $\Phi_{A,B} \circ \Phi_{C,D}$. Then, $(X, Y) \in \mathrm{PCP}_d^2$. In particular, the (C)DUC map $\Phi_{A,B} \circ \Phi_{C,D}$ is entanglement breaking

- Consider the case of CDUC maps and $d = 3$. We have

$$Y = \begin{bmatrix} A_{11}C_{11} + A_{12}C_{21} + A_{13}C_{31} & B_{12}D_{12} & B_{13}D_{13} \\ B_{21}D_{21} & A_{21}C_{12} + A_{22}C_{22} + A_{23}C_{32} & B_{23}D_{23} \\ B_{31}D_{31} & B_{32}D_{32} & A_{31}C_{13} + A_{32}C_{23} + A_{33}C_{33} \end{bmatrix}$$

$$= \mathrm{diag}(A \odot C) + \begin{bmatrix} A_{12}C_{21} & B_{12}D_{12} & 0 \\ B_{21}D_{21} & A_{21}C_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} A_{13}C_{31} & 0 & B_{13}D_{13} \\ 0 & 0 & 0 \\ B_{31}D_{31} & 0 & A_{31}C_{13} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{23}C_{32} & B_{23}D_{23} \\ 0 & B_{32}D_{32} & A_{32}C_{23} \end{bmatrix}$$

- For arbitrary d, we have

$$Y = \mathrm{diag}(A \odot C) + \sum_{1 \le i < j \le d} \begin{bmatrix} A_{ij}C_{ji} & B_{ij}D_{ij} \\ B_{ji}D_{ji} & A_{ji}C_{ij} \end{bmatrix}_{\mathbb{C}|i\rangle \oplus \mathbb{C}|j\rangle}$$

proving that Y has factor width 2

The take-home slide

- Linear maps (and bipartite matrices) covariant under the action of the diagonal unitary group: $\forall U \in \mathcal{DU}_d$, $\Phi(UXU^*) = U\Phi(X)U^*$
- $\Phi_{A,B}(X) = \operatorname{diag}(A\,|\operatorname{diag} X\rangle) + (B - \operatorname{diag} B) \odot X$
- $\Phi_{A,B}$ is CP $\iff$ $A \in \mathrm{EWP}_d$ and $B \in \mathrm{PSD}_d$. It is PPT if, moreover, $A_{ij}A_{ji} \geq |B_{ij}|^2 \quad \forall i,j$. It is EB $\iff$ $(A,B) \in \mathrm{PCP}_d$:

$$A = \sum_n |v_n \odot \overline{v_n}\rangle\langle w_n \odot \overline{w_n}| \qquad B = \sum_n |v_n \odot w_n\rangle\langle v_n \odot w_n|$$

- Main result: PPT^2 holds for (C)DUC maps

- Most of the above can be extend to linear maps which are covariant with respect to the diagonal orthogonal group $\mathcal{DO}_d \rightsquigarrow$ DOC maps
- $\Phi_{A,B,C}(X) = \operatorname{diag}(A\,|\operatorname{diag} X\rangle) + (B - \operatorname{diag} B) \odot X + (C - \operatorname{diag} C) \odot X^\top$
- $\Phi_{A,B,C}$ is EB $\iff$ $(A,B,C) \in \mathrm{TCP}_d$ [SN20a]:

$$A = \sum_n |v_n \odot \overline{v_n}\rangle\langle w_n \odot \overline{w_n}| \quad B = \sum_n |v_n \odot w_n\rangle\langle v_n \odot w_n| \quad C = \sum_n |v_n \odot \overline{w_n}\rangle\langle v_n \odot \overline{w_n}|$$

- No simple criterion for memb. in $\mathrm{TCP}_d \rightsquigarrow$ PPT^2 is open for DOC maps

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